Helping Behavior in Networked Organizations

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Abstract

This article studies a networked organization in which agents work to directly increase the team’s output and help neighbors to reduce the disutility of working. I build a tractable framework to analyze the complexity of mutual help that cannot be simply characterized as strategic complements or substitutes, and the relationship between the structure of social networks within an organization and its efficiency. I establish the existence and uniqueness of the equilibrium of a two-stage game in which agents first decide how much helping effort to give to each neighbor, and, then, how much effort to expend to directly benefit the team. I show that denser networks might not necessarily sustain a higher level of help, because links might be redundant. If the agents are homogeneous, the network in which they are pairwise connected reaches the highest efficiency with the fewest links.

JEL Classification: D23, D85, L23

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1 Introduction

In this paper, I study the relationship between organizational structure and production efficiency, in which the organization features arbitrary interpersonal connections or networks. Such a relationship is one of the central issues in organizational economics. However, due to the complexity of the organizational structure, especially the structure of social networks within the organization, few papers have made in-depth theoretical investigations in this area. In this paper, I construct a novel and tractable framework to analyze this issue, and the framework can also be exploited to shed light on other related issues, such as the optimal compensation scheme in networked organizations.

It is well-established that the well-functioning of an organization depends not only on each agent’s full engagement in their own job, but also on effective teamwork or cooperation. This is especially true when the output of the organization is a complementary combination of agents’ own efforts and when each agent’s compensation hinges only on the team’s output and not on their own contribution (due to the lack of separability or observability of individual contributions). In the absence of mutual help, or more broadly, peer effect, agents do not have enough incentive to work hard, in that they bear the full cost of their effort but gain only $1/N$ of the output (in an $N$-agent organization), as Kandel and Lazear (1992) point out. On the contrary, Kandel and Lazear (1992) argue that if each agent can help (or monitor) their peers, then the helped agents are motivated to work harder. This, in turn, increases the team’s output and the compensation received by the agent who offers help.

Mutual help may be more likely to occur among workers who have a better relationship, in the sense that they have a smaller psychological cost to help one another: they may be friends or have similar social backgrounds. Such fact is supported by psychological studies (Shapiro, 1980). However, the relationship between the connectedness of the social network and the team’s efficiency is ambiguous. Consider two extreme cases. On the one hand, when each agent is isolated or, in other words, has no connection to others, no one would readily help, which is surely not a desirable situation. On the other hand, however seemingly counter-intuitive, the overall efficiency is not necessarily high when everyone has many connections. This is because agents know that their friends would be helped

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1Help reduces the marginal disutility of working, whereas monitoring increases the marginal cost of shirking. In this article I only discuss mutual help, which is “positive peer pressure” in Calvó-Armengol and Jackson (2010).

2A measure of such connections is hometown connection, as is illustrated in Bandiera, Barankay, and Rasul (2009).

3In the model, I assume that help can only be provided between connected agents, acquaintances, or friends. More discussions can be found in the Model Section.
by their friends’ friends, so instead of providing helping effort on their direct friends, agents would rather free-ride on indirect friends’ help. Thus, the structure of the social network is essential for the performance of an organization, and its effects on performance need careful study.

In this article, I provide the first analysis of the relationship between the connectedness of the social network within an organization and the efficiency of the organization. I build a novel and tractable framework that departs from the classical literature that exploits quadratic preferences and linear best-responses to analyze the complexity of helping behavior. I study a model in which a team of agents work collectively to produce the team’s output, which the agents share equally. In addition to providing their own effort, which contributes directly to the team’s output, each agent can also help other agents to reduce the marginal disutility of providing their own effort and, hence, encourage them to expend greater effort. As a result of the help, the team’s output rises, and, thus, the compensation received by the agent who offers help also increases. I further assume that each agent possesses some connections to other agents and that each agent is willing to give help to another agent if and only if the two agents are connected. Thus, the structure of the social network matters for the operation of mutual help and the efficiency of the organization. In the game, agents first decide simultaneously how much helping effort to devote to the neighboring agents and then how much of their own effort to expend. Helping efforts reduce the marginal disutility of the agents’ own efforts, and agents must help each other first and then provide their own effort (taking into account the disutility established on the helping efforts). I characterize the subgame perfect Nash equilibrium of this non-cooperative game for special functional forms, thus building a tractable framework for a complicated game-theoretic setting.

The model setting yields a non-linear best response function (same as in Bramoullé and Kranton 2007 and Bourlès, Bramoullé, and Perez-Richet 2017) for each agent but is still tractable. The tractable setting, or framework, to analyze mutual help in networked organizations, is one major contribution of this paper. I investigate the multiplicity of the equilibria. I show that, although different equilibria exhibit different patterns of mutual help, all equilibria lead to the same profile of the agent’s own effort, and each agent gives and receives the same total amount of help across different equilibria. This is the weak uniqueness of the equilibrium. Different than the existing literature that

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4I discuss the relationship with the existing literature in Section 2.

5For the sake of brevity, I will use the term “their own effort” or “the agent’s own effort” to refer to agents’ efforts that directly increase the team’s output, as opposed to helping neighbors to reduce the disutility of working, which indirectly increases the team’s output.
uses the property of maximum/minimum eigenvalue, I use a special version of the fixed-point theorem (Theorem 3 in [Kennan 2001]) to establish the weak uniqueness of equilibrium. I also define and construct an auxiliary matrix, and I find that the strict uniqueness depends on its rank. Moreover, under the output sharing scheme, since there is a free-riding problem for exerting own effort and helping effort, the equilibrium cannot achieve the first-best outcome.

My analysis emphasizes the externality of help. In equilibrium, an agent, say \( i \), may have the incentive to help a neighboring agent, \( j \), because by doing so, \( i \) can increase the team’s output and, thus, his or her own compensation. Yet the effects of the help on others are more complicated. Agent \( i \)’s helping \( j \) has two impacts. On the one hand, as the model assumes, the marginal benefit of help is decreasing, \( i \)’s helping \( j \) lowers the marginal benefit for others to help \( j \) and, hence, discourages \( j \)’s other neighbors from helping \( j \). In this respect, the decisions about helping effort by different agents are strategic substitutes. On the other hand, however, due to the complementarity of the team’s production function, \( i \)’s helping \( j \) increases \( j \)’s own effort and further increases the marginal products of all the other agents’ own effort. Consequently, it encourages all agents to help their neighbors. In this respect, decisions on helping effort are strategic complements.

Therefore, the equilibrium outcome is affected by the two impacts of the externality of help, implying that the effect on the team’s output of adding new connections to the organization network is ambiguous. A well-connected organization may not necessarily outperform others because it would have many redundant links that cannot improve performance. To be more specific, adding links to the organization only weakly increases its performance. If the organization has an even number of agents, then the network that has the fewest links among ones that maximize the team’s output in equilibrium should be one in which agents are pairwise linked; if the organization has an odd number of agents, the most efficient network should be one in which \( N - 3 \) agents are pairwise linked, and the other three agents are linked in a circle (suppose an \( N \)-agent organization). The intuition is that while each agent must be helped by at least one other agent to provide stronger incentives for the agent’s own effort, (s)he should not be helped by strictly more than one, since the return of providing helping efforts is decreasing and the helping effort provided by different agents crowds out one another. Thus, each agent being helped by exactly one other agent perfectly balances all the benefits and costs of helping effort and, thus, is optimal.\(^6\)

\(^6\)The result of the optimal organizational structure justifies the existence of couple jobs. Consider the extreme case, in which a farm only hires couples and each couple does not know other couples. Therefore, several couples work together as a team for the farm, but there are no connections between couples. There is, however, a
The remainder of the article is organized as follows. Section 2 reviews the related studies. Section 3 introduces the model. Section 4 proves the existence and uniqueness of the subgame perfect Nash equilibrium, provides several illustrative examples, and analyzes the comparative statics. Section 5 concludes.

2 Literature Review

This article speaks to two strands of the literature. First, this article introduces social networks to the study of a specific peer effect—mutual help—in an organization. Thus, it extends the seminal work of Kandel and Lazear (1992), who consider the peer effect (in their terms, peer pressure) as a means to internalize the externality of shirking. Peer pressure works as a psychological cost if the agent being pressured shirks. In an organization that uses output sharing as its compensation scheme, peer pressure can incentivize each agent to work harder and, hence, improve the overall efficiency of the organization. In their article, however, the impact of peer pressure is specified simply via an exogenous reduced-form “peer pressure” function, in which the peer pressure felt by each agent decreases with his or her own effort. Therefore, the mechanism of how peer pressure works needs further illustration. Moreover, there is no interpersonal connection in Kandel and Lazear (1992). Hence, I focus on how the network structure shapes the specific pattern of peer pressure, which is a major difference from Kandel and Lazear (1992). In addition, my article is related to Calvó-Armengol and Jackson (2010), who consider a broader range of peer pressure. The differences between this paper and Calvó-Armengol and Jackson (2010) are: (1) the strategies in this paper are continuous, and those in Calvó-Armengol and Jackson (2010) are discrete; (2) there is a network among peers in this paper whereas the network does not exist in Calvó-Armengol and Jackson (2010); (3) the strategies cannot be simply classified as strategic complements or substitutes in this paper, while they can in Calvó-Armengol and Jackson (2010).

Second, my analysis is built on the general literature on complete-information games played on fixed networks (Ballester, Calvó-Armengol, and Zenou, 2006; Ballester, Zenou, and Calvó-Armengol, 2006). This is a pairwise-connected network as in the theoretical prediction. Examples of couple job postings in farms can be found as follows: https://www.ziprecruiter.com/Jobs/Farm-Couple; https://www.indeed.com/q-Farm-Couple-jobs.html?vjk=6e0ab36c53db7f5. Couples are better able to help each other, and, thus, raise the productivity of the entire organization.

Alchian and Demsetz (1972), Itoh (1991), Rotemberg (1994), Drago and Garvey (1998), Rayo (2007), and Battiston et al. (2022) also discuss peer effects or moral hazard in teams and using monitoring to reduce such moral hazard, but they are also silent on the internal structure of the team or the organization.
However, my analysis is different from the cited articles in two respects. One is that, in my analysis, the relationship between the strategies of different agents is complicated, and, thus, it cannot simply be classified as strategic complements or substitutes, as in these articles. The other is that I analyze a model in which agents’ strategies are multidimensional and their best-replies are non-linear. Most of the literature considers only single-dimensional strategies, quadratic preferences, and linear best-reply functions, with the exceptions of \cite{Bramoule2007} and \cite{Bourles2017}. My work resembles \cite{Bourles2017} the most in the sense that we both have multi-dimensional strategies and non-linear best-replies in the model. In my analysis I use special functional forms, especially the Cobb-Douglas team production functions, to simplify the calculation and, thus, make the model tractable. Constructing a tractable mathematical framework for the case of team production with network interaction is also one of the major contributions of this article. Moreover, different from the existing literature on network games that establishes the existence and uniqueness of equilibrium, the best-reply function in this model is non-linear and, thus, cannot use the property of the smallest or the largest eigenvalue. Instead, I use a special version of the fixed-point theorem, as in \cite{Kennen2001} to establish the weak uniqueness of equilibrium. I also define and construct an auxiliary matrix, and I find that the uniqueness depends on its rank, which is a spectral property due to the rank-nullity theorem.

3 Model

In the model, there is a team of agents who work collectively to produce a team’s output. In addition, each agent can help other agents to reduce their marginal disutility of working and, hence, encourage them to work harder; as a result, the team’s output would increase. Each agent possesses some connections to other agents, and he or she can give help to an agent if and only if he or she and...
that agent are connected. Thus, the model extends the traditional principal-multiagent framework in two respects: (1) there are social connections among agents—this is the distinction between this article and Kandel and Lazear (1992); and (2) each agent chooses not only his or her own effort that contributes to the team’s output but also how much effort should be devoted to helping every agent to whom he or she is connected. The details of the model are as follows.

**Network among Agents** The model consists of a team of \( N \geq 2 \) agents, denoted by \( A = \{1, 2, ..., N\} \). Agents work together to produce a team’s output, taking the equal-sharing compensation scheme set as given. Different from the existing (principal-)multi-agent models, here, it is assumed that agents are interconnected. Connections between agents could be seen as some interpersonal relationships such as acquaintance or friendship. Agents connected to \( i \) are deemed their “neighbors,” whose set is denoted by \( N(i) = \{j \in A : g_{ij} = 1\} \). The pattern of connections among agents is represented by an \( N \times N \) matrix \( G \), which represents the adjacency of agents. Each entry in \( G \), \( g_{ij} \in \{0, 1\} \), denotes whether there is a connection between \( i \) and \( j \), and \( g_{ij} = 1 \) means that \( i \) and \( j \) are connected. In the model, I assume that connections are mutual, or undirected. However, the main results still hold for directed connections. By convention, \( g_{i,i} = 0 \).

In the following part of this paper, I first study the equilibrium given any fixed network structure; then I study which network structure achieves the highest efficiency, and additionally introduce the cost of maintaining links. I formally define an overall efficiency (Definition 2 in Section 4.3) which is equal to the team’s output net of the total fixed costs of forming and maintaining connections, which are incurred before the strategic interactions among agents.

As illustrated in detail below, I assume that only connected agents can, or, to be more specific, are willing to help each other. Therefore, the role of social networks in this model is to enable and foster help and cooperation. Such an assumption is based on a consensus in the psychological literature that mutual help is much more commonly observed between friends than between strangers (Shapiro, 1980). An equivalent assumption is that the cost of helping a non-neighbor agent is infinite. I can also relax the assumption to the case that the cost of helping a non-neighbor agent is finite but strictly higher than that of helping a connected agent. In the case of such an extension, most of the results (all the following propositions and existence and weak uniqueness of the equilibrium) still hold. \(^9\)

\( ^9 \)The network is undirected and connections are mutual, i.e., \( g_{i,j} = g_{j,i} \) and \( G \) is a symmetric matrix.

\( ^{10} \)The above setting is equivalent to the following: assuming that in principle, each agent can help anyone in the team (the ability to help). However, if two agents are not connected, then the cost of help is sufficiently large (even infinite) so that no one is willing to do it (the willingness to help). This is consistent with the previous study in psychology, Shapiro (1980), that mutual help occurs much more frequently among friends or
Agents’ Own Effort and Helping Effort  In the model, the ability of own effort $\theta$ and the ability of helping effort $\gamma$ are common knowledge. Each agent $i \in A$ chooses his or her own level of effort, which is denoted by $e_i$, as well as the level of helping effort to each neighbor $j \in N(i)$, which is denoted by $h_{ij}$. I assume that the marginal cost of helping non-neighbors is sufficiently large that each agent is not willing to help any other agents except her neighbors. This is equivalent to assuming that each agent “can only afford to” help their neighbors\footnote{To be more specific, the cost of helping non-neighbors is $\frac{1}{2}\bar{\gamma}(\sum_{j \notin N(i)} h_{ij})^2$, where $\bar{\gamma}$ is a number that is sufficiently large that prevents helping. Here I assume that the cost takes a quadratic functional form.} The collection of all agents’ own efforts is denoted by $E = \{e_1, \ldots, e_N\}$, and the collection of the profile of helping effort of all agents is denoted by an $N$ by $N$ matrix $H$, in which each entry, $h_{ij}$, is how much helping effort $i$ provides to $j$. In addition, the helping effort each agent receives is denoted by $H^r = \{\sum_{k \in N(1)} h_{1k}, \ldots, \sum_{k \in N(N)} h_{kN}\}$, and the helping effort given is denoted by $H^g = \{\sum_{k \in N(1)} h_{1k}, \ldots, \sum_{k \in N(N)} h_{Nk}\}$. Assume, here, that each agent has limited time and energy to expend on the two types of efforts; that is, both the agent’s own effort and the sum of helping effort have an upper bound, and, hence, $e_i \in [0, \bar{e}]$, and $\sum_{j \in N(i)} h_{ij} \in [0, \bar{h}]$, where $\bar{e}, \bar{h} > 0$ are fixed constants and sufficiently large so that, in equilibrium, each agent’s strategy can never reach the upper bound. In this setting, the strategy space for each agent is a convex and compact set. Such a setting is relevant for the existence and weak uniqueness of the equilibrium.

Agents’ own effort and helping efforts can be understood in the following way. Consider a group of workers on a farm, who are responsible for growing fruit. Each worker’s own effort is to grow as much fruit as (s)he can that contributes directly to the farm’s outputs. At the same time, each agent can also help other agents, like handing over water to those who are thirsty or sharing his/her own food with those who are hungry. Such helping effort does not directly contribute to the amount of fruit, or the farm’s output, but reduces the marginal disutility of working.

Each type of effort induces disutility or cost to the agents. The cost of $i$’s own effort is $(\theta + \sum_{j \in N(i)} h_{ji})^{-\beta} e_i$, where $e_i$ is the level of $i$’s own effort, and $(\theta + \sum_{j \in N(i)} h_{ji})^{-\beta} \equiv c(\theta, \sum_{j \in N(i)} h_{ji})$ is the marginal disutility of exerting the agent’s own effort, which is affected by $i$’s “ability” to exert own effort, $\theta$, and by the helping effort received from $i$’s neighbors, $\sum_{j \in N(i)} h_{ji}$. There are four underlying assumptions here: (1) help among agents takes effect via reducing the marginal disutility of the agent’s own effort; (2) helping efforts received from different neighbors are additively separable; acquaintances than among strangers. But still, consistent with this study, the connections in the network can be seen as friendship or acquaintanceship that allows for willingness to help. Finally, since no one is willing to help non-neighbors, the cost of doing so can be omitted from the payoff function.
(3) the effects of the helping effort provided by different neighbors are identical; and (4) the “ability” to exert the agent’s own effort or the marginal disutility of exerting effort without any help, $\theta$, are identical across agents. In addition, in this setting, $c(\cdot)$ is a smooth function with $c \geq 0$, $c'_1 = -\beta(\theta + \sum_{j \in N(i)} h_{ji})^{-\beta - 1} \leq 0$, $c'_2 = -\beta(\theta + \sum_{j \in N(i)} h_{ji})^{-\beta - 1} \leq 0$, and $c''_{22} - \beta(\beta + 1)(\theta + \sum_{j \in N(i)} h_{ji})^{-\beta - 2} \geq 0$, which means that the marginal effect of helping effort is decreasing. Disutility of helping effort is denoted by $\gamma v(\sum_{j \in N(i)} h_{ij})$, where $\gamma$ denotes the “ability” to help others, and $v$ is a smooth function with $v \geq 0$, $v' \geq 0$, $v'' \geq 0$. For simplicity, assume that $v(x) = 0.5x^2$. Hence, the marginal disutility of helping effort at zero is zero, which guarantees that, in equilibrium, all agents are willing to help.

An important setting in the model is that agents provide helping effort and their own effort separately, so the cost of these two types of effort enters the payoff function separately. I make this assumption for the tractability of the model. If this assumption is not made, then the complexity of the relationship between own effort and helping effort makes the model highly untractable.

### Production Technology, Compensation, and Payoff

Team output $y$ is a function of all agents’ own efforts: $y = f(e_1, ..., e_N)$. Assume here that $f$ takes the Cobb-Douglas functional form: $f(e) = \prod_{i=1}^{N} e_i^{w_0}$, where $Nw_0 < 1$. Thus, each agent plays an identical role in production, and the production function exhibits a decreasing return to scale. In addition, $f$ is a smooth function with strict concavity, $f \geq 0$, $f'_i = w_0 e_i^{-1} \prod_{i=1}^{N} e_i^{w_0} \geq 0$ and $f''_{ij} = w_0^2 e_i^{-1} e_j^{-1} \prod_{i=1}^{N} e_i^{w_0} \geq 0$, for any $i$ and $j$, which means that the production technology exhibits the property of complementarity. The assumption of strict concavity is essential for the existence and uniqueness of the equilibrium.

The compensation scheme depends only on the team’s output $y$. Consistent with the former literature [Kandel and Lazear, 1992], here, the analysis considers only the simple output sharing rule, where a share of $N\bar{\alpha} \in (0, 1)$ of the output as total compensation is divided and distributed equally among the agents. Hence, each agent gets $\bar{\alpha}y$. We assume that $\bar{\alpha}$ is exogenously set (by a hypothetical principal or by the unanimous agreement of all agents).

The payoff of each agent is the compensation obtained minus the disutility from exerting his or her own effort and helping effort. Therefore, the payoff of agent $i$, given the above output sharing

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12Hence, $v' = x$ and $v'' = 1$.

13The form of Cobb Douglas makes the model tractable. More general functional forms, such as the Constant Elasticity of Substitution, do not.

14Since I study the case in which the compensation scheme is exogenously set, I ignore the role of a principal in the model. All agents share the team’s output equally. There are many other schemes of compensation, but in this article, as I do not study the optimal compensation scheme (which, may be set by a principal), and I assume that the compensation scheme is set exogenously and the organization adopts a simple equal-sharing scheme for ease of analysis.
compensation and functional forms, can be written as
\[
\bar{\alpha} \prod_{i=1}^{N} e_i^{\rho_0} - (\theta + \sum_{j \in N(i)} h_{ji})^{-\beta} e_i - \frac{1}{2} \gamma (\sum_{j \in N(i)} h_{ij})^2.
\]

**Timing of the Game** The game proceeds as follows. In stage 0, the compensation scheme \(\bar{\alpha}\) is exogenously set. This article does not study the principal’s optimal strategy—that is, how to set the optimal portion for output sharing; hence, \(\bar{\alpha}\) is set as an exogenous and fixed constant. In stage 1, given the team’s network structure \(G\) and the compensation scheme \(\bar{\alpha}\), each agent simultaneously chooses his or her helping efforts \(H_i = (h_{i1}, \ldots, h_{ij}), j \in N(i)\). In stage 2, given the team’s network structure \(G\), the compensation scheme, and helping efforts \(H\) chosen in the former stage, each agent simultaneously decides his or her own effort \(e_i\). This setting is the same as that of Calvó-Armengol and Jackson (2010), where agents first choose peer pressure exerted on other agents and then choose another action (such as whether to participate) based on former choices of peer pressure. Helping efforts reduce the marginal disutility of the agents’ own efforts, and agents must help each other first and then exert their own effort (taking into account the disutility established on the helping efforts).

In this article, I focus on the (pure strategy) subgame perfect Nash equilibria of the game, in which agents first decide the pattern of mutual help devoted to their neighbors and then decide their own efforts. Variables with a superscript * denote those in equilibrium.

4 Equilibrium

In this section, I analyze whether there exists any subgame perfect Nash equilibrium in the above game and examine the basic properties of the equilibrium. Recall that I set the compensation scheme, \(\bar{\alpha}\), fixed, and I analyze only how agents interact with one another. The two-stage game always has a trivial equilibrium: all agents provide zero helping efforts in the first stage, and then exert zero own effort in the second stage. In this equilibrium, the output is zero and, thus, is certainly Pareto inferior. I do not focus on this equilibrium hereafter. All the proofs are relegated to Appendix A.

4.1 Equilibrium Existence

First, I provide the proposition for the existence of a subgame perfect Nash equilibrium, for the specific functional forms of the model. The proposition on the existence of the equilibrium is as follows:
**Proposition 1.** Given the output sharing rule and the above-mentioned functional forms, a non-trivial pure-strategy subgame perfect Nash equilibrium of the game exists.

This is a direct implication of Theorem 1 in [Rosen (1965)](https://ssrn.com/abstract=3687345), which states that in an \( N \)-person concave game where the strategy space of each agent is convex and compact, and the payoff function is concave in the agent’s own strategy, there always exists a Nash equilibrium. The overall game has two (generic) subgames starting from stages 1 and 2, and it can be verified that the existence condition stated by Theorem 1 in [Rosen (1965)](https://ssrn.com/abstract=3687345) is satisfied in these two generic classes of subgames.

I use backward induction to solve for the equilibrium. In stage 2, given the collection of helping effort in stage 1 and the choices of others’ own effort, each agent decides the level of her own effort to maximize her payoff, taking the choices of other agents \( \{e_j^*\}_{j \neq i} \) as given. Hence,

\[
e_i^* = \arg \max_{e_i} \bar{\alpha} e_1^* w_0 e_2^* w_0 \cdots e_i^* w_0 \cdots e_N^* w_0 - (\theta + \sum_{j \in N(i)} h_{ji})^{-\beta} e_i - \frac{1}{2} \gamma (\sum_{j \in N(i)} h_{ij})^2. \tag{1}
\]

The strategy space for \( e_i \) is convex and compact, and the objective function is concave in \( e_i \); therefore, there exists a Nash equilibrium of this subgame.

The first-order condition for agent \( i \) with respect to \( e_i \) is that

\[
\bar{\alpha} w_0 e_1^* w_0 e_2^* w_0 \cdots e_i^* w_0^{-1} \cdots e_N^* w_0 - (\theta + \sum_{j \in N(i)} h_{ji})^{-\beta} = 0,
\]

for each \( i \in 1, 2, ..., N \).

Combining the \( N \) first-order conditions for each agent, the agent’s own equilibrium effort of the subgame of stage 2 can be solved. Given the specific functional form, the system of equations has a unique and strictly positive solution \( \{e_i^*\} \):

\[
e_i^* = (\bar{\alpha} w_0)^{\frac{1}{\beta w_0}} (\theta + \sum_{j \in N(i)} h_{ji})^\beta \prod_{j=1}^{\beta w_0} (\theta + \sum_{k \in N(j)} h_{kj})^{\frac{\beta w_0}{\beta w_0}} \tag{2}
\]

By backward induction, substituting equation (2) into the payoff function, in stage 2, agent \( i \)'s problem is to find \( \{h_{ij}\}_{j \in N(i)} \) given \( \{h_{kl}^*\}_{k \neq i, j \in N(k)} \):

\[
h_{ij}^* = \arg \max_{h_{ij}} (\bar{\alpha} w_0)^{\frac{1}{\beta w_0}} (\frac{1}{w_0} - 1)(\prod_{k=1}^{N} (\theta + \sum_{l \in N(k)} h_{lk}^*)^{\frac{\beta w_0}{\beta w_0}} - \frac{1}{2} \gamma (\sum_{j \in N(i)} h_{ij})^2). \tag{3}
\]
and the first-order condition with respect to \( h_{ij} \) can be derived as an inequality as follows:

\[
\frac{\beta w_0}{1 - N w_0} \prod_{j \geq 1} (\theta + \sum_{k \in N(j)} h_{kj}^*) j - N w_0 (\theta + \sum_{k \in N(j)} h_{kj}^*)^{-1} (\bar{\alpha} w_0) j - N w_0 (\frac{1}{w_0} - 1) \leq \gamma (\sum_{j \in N(i)} h_{ij}^*) \tag{4}
\]

The first-order condition \((1)\) depicts the relationship between the total helping effort provided by \( i \), \( \sum_{j \in N(i)} h_{ij}^* \), and the total effort received by \( j \), \( \sum_{k \in N(j)} h_{kj}^* \). If \( h_{ij} \) takes a corner solution, then \((1)\) takes a strictly less than equal sign. If \( h_{ij} \) takes an interior solution, then \((1)\) takes an equal sign. If \( h_{ij} = 0 \), \((1)\) can take a strict less than equal sign or an equal sign. Also, \( h_{ij} = 0 \) can be an interior solution as well as a corner solution. If neither \( h_{ij} \) nor \( h_{ji} \) takes a non-zero interior solution, then link \( ij \) is said to be redundant. The details of mathematical derivation and algebraic manipulation are relegated to Appendix A.

In addition, a simple manipulation of equation \((1)\) shows that

\[
(\sum_{j \in N(i)} h_{ij}^*) (\theta + \sum_{k \in N(j)} h_{kj}^*) \geq \frac{\beta w_0}{1 - N w_0} \prod_{j \geq 1} (\theta + \sum_{k \in N(j)} h_{kj}^*) \frac{\beta w_0}{\bar{\alpha} w_0} (\bar{\alpha} w_0) - N w_0 (\frac{1}{w_0} - 1) \gamma^{-1} \equiv M(H^*), \tag{5}
\]

where \( M(H^*) \) is identical for all agents. In equilibrium, if there exist agents \( i,j,l,m \in A \) such that \( h_{ij}^* \) and \( h_{kl}^* \) take interior solutions, then \((\sum_{j \in N(i)} h_{ij}^*) (\theta + \sum_{k \in N(j)} h_{kj}^*) = (\sum_{j \in N(i)} h_{ij}^*) (\theta + \sum_{k \in N(m)} h_{km}^*) \), which, thus, provides an important equation for computing the equilibrium. From the above algebra, we can derive the following proposition, which can help us understand the mechanisms that shape the equilibrium. Specifically, equation \((1)\) directly implies Proposition \(2\) as follows:

**Proposition 2.** From the first-order conditions, we have the following three results: (1) In equilibrium, each non-isolated agent (who has at least one neighbor) receives and gives a strictly positive amount of help. (2) In equilibrium, each agent helps only the neighbors who receive the least amount of help (there would be more than one neighbor receiving the minimum help). In addition, only neighbors who give the least amount of help would help each agent (there would be more than one neighbor giving the minimum help). (3) In equilibrium, for each agent \( i \), if there are two neighbors \( j_1, j_2 \) such that \( h_{j_1}^* \) and \( h_{j_2}^* \) take interior solutions, then \( j_1 \) and \( j_2 \) give the same amount of help. Similarly, if \( h_{ij_1}^* \) and \( h_{ij_2}^* \) take interior solutions, then they receive the same amount of help.

To express Proposition \(2\) mathematically, it is equivalent to saying that for any \( j \in N(i) \), \( h_{ij}^* > 0 \) if \( \sum_{k \in N(j)} h_{kj}^* \leq \sum_{k' \in N(j')} h_{k'j'}^* \) for all \( j' \in N(i) \) (each agent helps only the neighbors who
receive the least amount of help), and that for any $i \in N(j)$, $h_{ij}^* > 0$ if $\sum_{k \in N(i)} h_{ik}^* \leq \sum_{k' \in N(i')'} h_{ik'}^*$ for all $i' \in N(j)$ (only neighbors who give the least amount of help would help each agent). As the marginal cost of the helping effort is zero when the total helping effort is zero, but the marginal return is strictly positive, all agents are willing to help. On the other hand, in the network structure of the organization, all agents balance the marginal return and the marginal cost of helping their different neighbors and, hence, will not leave any of their neighbors without help. Mathematically, it is equivalent to comparing the complementary slackness (first-order) conditions to establish the corresponding inequality. These lead to part (1). In the network, agents try to prioritize their help to neighbors with the highest marginal return. These neighbors are those who receive the minimum level of help. On the other hand, agents whose marginal cost of helping is the smallest have the strongest incentive to help others. These lead to part (2). Finally, if the first-order condition, equation (4), takes an equal sign for two neighbors of the same agent, then we can equate the total given or received helping effort. This leads to part (3).

The above Proposition 2 sheds light on how to compute the equilibria for any given network, using the following four steps. Step 1: Assuming that all helping efforts are strictly positive, use Proposition 2(3) to find which nodes give and receive an equal amount of helping effort. This step gives $4l - 2n$ equalities for the total given and received helping efforts of neighboring agents. Step 2: Given the set of equations in step 1, helping efforts that are not strictly greater than zero can be found. These helping efforts identify redundant links. The helping efforts associated with the redundant links are zero, and the related equalities for the total given and received helping efforts of neighboring agents do not hold. Step 3: Disregard all the redundant links, and, thus, the whole graph is cut into several small subgraphs. For each subgraph, repeat steps 1 and 2 until all helping efforts in the non-redundant links are strictly positive. Step 4: Express the total given and received helping efforts as a linear function of the remaining unknowns, and apply the first-order condition (4).

Proposition 3 and Corollary 1 further show that the equilibria can be characterized by bipartite/non-bipartite graphs. A bipartite graph is a graph where the vertices can be divided into two disjoint sets such that all edges connect a vertex in one set to a vertex in another set. Such a set is an independent set of the bipartite graph.

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15This step is completed given the network structure $G$ that determines which pairs of agents are not linked and, thus, cannot help each other.
16With $n$ nodes and $l$ links, there are $2l - n$ pairs of neighbors, and, thus, $2(2l - n) = 4l - 2n$ equations for total helping efforts.
17Example B1 shows how to identify any redundant links.
Proposition 3. Suppose that $H^*$ is an equilibrium profile of helping efforts in which the first-order conditions of all elements take interior solutions. Given $H^*$, define a new matrix $\bar{H}^* = (\bar{h}_{ij}^*)$ whose entries are zeros or ones, with $\bar{h}_{ij}^* = 0$ if $h_{ij}^*$ in $H^*$ takes a corner zero solution, and $\bar{h}_{ij}^* = 1$ if $h_{ij}^*$ in $H^*$ takes an interior solution. Then, if $\bar{H}^*$ is a bipartite graph, then there are no more than two different amounts of the total received helping efforts ($\sum_{k \in N(j)} h_{jk}^*$) and two different amounts of the total given helping efforts ($\sum_{k \in N(j)} h_{kj}^*$), and agents in each of the two independent sets give and receive, respectively, the same amount of helping efforts.

To rephrase Proposition 3 mathematically, the argument that “there are no more than two different amounts of the total received helping efforts and two different amounts of the total given helping efforts ($\sum_{k \in N(j)} h_{jk}^*$), and agents in each of the two independent sets give and receive, respectively, the same amount of helping efforts” is equivalent to saying that the elements in each of the sets $\{\sum_{k \in N(j)} h_{jk}^*\}_{j \in A}$ and $\{\sum_{k \in N(j)} h_{kj}^*\}_{j \in A}$ take at most two distinct values. Moreover, if the bipartite graph $A = A_1 \cup A_2$, where $A_1$ and $A_2$ are two independent sets, then for all $j_1, j_2 \in A_k$ ($k = 1, 2$), $\sum_{k \in N(j_1)} h_{kj_1}^* = \sum_{k \in N(j_2)} h_{kj_2}^*$, and $\sum_{k \in N(j_1)} h_{j_1 k}^* = \sum_{k \in N(j_2)} h_{j_2 k}^*$.

Corollary 1. If $\bar{H}^*$ is not a bipartite graph, then there is only one amount of total received or given helping effort.

Mathematically, Corollary 1 indicates that the elements in each of the sets $\{\sum_{k \in N(j)} h_{jk}^*\}_{j \in A}$ and $\{\sum_{k \in N(j)} h_{kj}^*\}_{j \in A}$ take only one distinct value. Moreover, for all $j_1, j_2 \in A$, $\sum_{k \in N(j_1)} h_{kj_1}^* = \sum_{k \in N(j_2)} h_{kj_2}^*$, and $\sum_{k \in N(j_1)} h_{j_1 k}^* = \sum_{k \in N(j_2)} h_{j_2 k}^*$.

Proposition 3 is built on Proposition 2(3) and, they are a result of balancing the benefit and cost of helping effort. In a bipartite graph, agents in the same independent set have the same incentive to balance the cost and benefit of helping effort and to provide helping effort and, hence, give and receive the same total amount of helping effort. In a non-bipartite graph, there is only one type of agent and, thus, these agents give and receive the same total amount of helping effort.

4.2 Equilibrium Uniqueness

This section characterizes the most important property of the subgame Nash equilibria: all equilibria lead to the same profile of agents’ own effort $E^*$. In other words, in any equilibrium, each agent gives and receives the same total amount of help. Only the specific pattern of help—who gives help to whom or who receives help from whom—might be different across equilibria.
Proposition 4. For any (fixed) network structure $G$, all subgame perfect Nash equilibria in this network lead to the same profile of agents’ own efforts (not considering the trivial equilibrium in which all efforts are zero) $E^* = \{e_1^*, ..., e_N^*\}$. Each agent provides and receives the same total amount of helping effort in different equilibria in that fixed network. Each agent also gets the same payoff across different equilibria.

Proposition 4 establishes the “weak” uniqueness of the equilibrium with respect to the total amount of given and received helping efforts, given the fixed network, which holds for any structure of the network. The intuition of the proof exploits a version of the fixed-point theorem that concerns the concavity and monotonicity of the payoff function. According to Kennan (2001), given the concavity and monotonicity of the payoff function (and additional assumptions that can be checked to be satisfied in this setting), there is a unique fixed point: $E^* = \{e_1^*, ..., e_N^*\}$. However, only the profile of the total amount of helping effort given and received by each agent is uniquely determined; given the same profile of the total helping effort, the specific pattern of helping effort associated with each connection is not uniquely determined.

In addition, the multiplicity of equilibria depends on the network structure: if there is no loop in the network, then the equilibrium helping effort can be uniquely determined. This is what Proposition 4 argues. Before we move to a more detailed characterization of the uniqueness of equilibrium, we first lay out the following lemma, which is in turn based on Definition 1.

Definition 1. An auxiliary matrix $AM(G)$ is uniquely determined by the network structure of $G$. Rewrite the matrix of helping effort $H = \{h_{ij}\}$ as a column vector $h = (h_k)^T$, with each $k$ corresponding to a unique $ij$ pair first increasing in $i$ and then increasing in $j$. Then, according to Proposition 2/3, we have a system of linear equations $AM(G)h = 0$. In particular, $AM(G)$ is a $2K \times M$ matrix in which $K$ is the number of rows of $G$ with more than one non-zero entry, and $M$ is the dimension of

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18Specifically, I use Theorem 3 in Kennan (2001). It is based on the condition that $f(\cdot)$ (in that paper) is increasing and strictly concave and satisfies $f(0) > 0$ (can be further relaxed to $f(0) \geq 0$, since we can focus on an additional fixed point other than zero), $f(a) > a$, and $f(b) < b$ for some vector $b > a > 0$. In my setting, since the $f(\cdot)$ function is Cobb-Douglas, $e(\cdot)$ function is linear in $e_i$, and $v(\cdot)$ is quadratic, the weakly positive (at zero), increasing, and strictly concave (given $Nw_0 < 1$) conditions are satisfied. Moreover, given the Inada conditions, given the marginal return of own effort $e_i$ goes to infinity as $e_i$ goes to zero (the function is sufficiently “steep”), $f(a) > a$ is satisfied with a vector $a$ that is close to 0, and given the marginal return of own effort $e_i$ goes to 0 (the function is sufficiently “flat” as $e_i$ goes to infinity, we can also find a vector $b$ that is sufficiently large (but still finite). Therefore, we can show that all conditions required by Theorem 3 of Kennan (2001) are easily satisfied in this paper’s setting.

19According to Rosen (1965), the uniqueness relies on the condition that $g(x, r)$ (pseudogradient of $\sigma(x, r)$) in that paper is diagonally strictly concave for some $r > 0$. Translating it to the setting of this paper, it is equivalent to requiring that the following vector $(r_1w_0e_1^{-1}\Pi_{i=1}^N e_i^{\theta w_0} - \theta + \sum_{j \in N(i)} h_{ij})^{-1} r_1, ..., r_Nw_0e_N^{-1}\Pi_{i=1}^N e_i^{\theta w_0} - \theta + \sum_{j \in N(i)} h_{ij})^{-1} r_N)$ is diagonally strictly concave for some vector $(r_1, ..., r_N) > 0$. Due to the specific functional forms, such a condition of diagonally strict concavity is too strong to be satisfied. Thus, it is not feasible to employ the uniqueness results in Rosen (1965) directly.
The details of constructing \( AM(G) \) from \( G \) are relegated to Appendix A.

**Lemma 1.** For a graph, \( G \), with \( N \geq 3 \) agents, \( AM(G) \) is of full rank if and only if \( G \) does not contain any loop.

Lemma 1 can be proved by induction. It builds a connection between the structure of the network and the spectral property of the auxiliary matrix of the network. It is also used to establish the following proposition.

**Proposition 5.** The equilibrium is (strictly) unique if \( AM(G) \) is of full rank. This corresponds to the case in which the network contains no loops. Otherwise, there are multiple equilibria (not considering the trivial equilibrium in which all efforts are zero).

The intuition for this result is that if there is no loop in the network (then the network is called a “tree”), the helping effort can be uniquely pinned down inductively from the leaf node to the upper-level nodes using first-order conditions, and finally to the root node; if there is a loop, then such an inductive procedure cannot be implemented. The property is also related to the spectral properties of \( G \) (or \( AM(G) \)), since according to the rank-nullity theorem, if \( AM(G) \) is not of full rank, then the nullity of \( AM(G) \) is strictly greater than 0. In other words, for an arbitrary \( L \times L \) submatrix of \( AM(G) \), \( S(AM(G)) \), where \( L = \min\{2K, M\} \), 0 is an eigenvalue of \( S(AM(G)) \).

This proposition is illustrated by example in the following subsection. Consider a circle of three agents; the total amounts of help given and received by each agent, \( \tilde{h} \), are uniquely determined, but there are many different \( H^* \)—that is, who gives help to (or receives help from) whom, as is depicted by Figure 1.

![Figure 1: Multiplicity of equilibria](image)

### 4.3 Comparative Statics

This section studies the comparative statics with respect to adding links between agents. To begin with, I show that adding links to a network only *weakly* increases the team’s output or performance.
In some cases, adding links does not change the team’s output. Consider the following proposition for the case of adding only one new link. The case of adding multiple links can be thought of as a case in which links are added sequentially and, thus, the proposition can be applied sequentially (the sequence does not matter).

**Proposition 6.** Adding a new link between two unconnected agents can only weakly improve team performance.

A redundant link is defined as that adding it to a network does not affect the equilibrium outcomes. While whether a link can be redundant or non-redundant can be hard to predict, the following two examples illustrate when the added link is redundant or non-redundant.

**Example 1. Adding a Link to a Bipartite Graph** Given any bipartite graph, adding a link between two nodes that belong to different independent sets, as in Figure 2(a), does not change the team’s performance; adding a link between two nodes that belong to the same independent set, as in Figure 2(b), strictly increases the team’s performance; however, as in the case in Figure 2(c), the team’s output does not change when a link between two nodes that belong to the same independent set is added.

![Figure 2: Adding Links in Bipartite Graphs](image)

Proposition 6 implies the following Proposition 7, which states the structure of the network that can attain the highest level of efficiency with the fewest links.

**Proposition 7.** Given that the team’s performance attains the highest level possible, the network that contains the fewest links is the following: (1) for $N = 2K$, the network is the one in which agents are pairwise linked; and (2) for $N = 2K + 1$, the network is the one in which $2K - 2$ nodes are pairwise linked, and the other three nodes are linked in a circle.

---

Without adding any links, Figure 2 shows a network of five agents that is a bipartite graph. Agents 1, 3, and 5 belong to one independent set, and agents 2 and 4 belong to the other.
The intuition is that while each agent must be helped by at least one other agent to provide stronger incentives for the agent’s own effort, (s)he should not be helped by strictly more than one, since the return of providing helping efforts is decreasing and the helping effort provided by different agents crowds out one another. Thus, each agent being helped by exactly one other agent perfectly balances all the benefits and costs of helping effort and, thus, is optimal.

In the above analysis, I do not consider the fixed cost of forming and maintaining connections, denoted by \( c_{ij}^F \) (the cost of connection between agent \( i \) and agent \( j \)). I assume that agents pay the cost prior to the network interaction stage and that their participation constraints do not bind (the payoff of their outside option is sufficiently negative).\(^{21}\) Instead, I focus on the structure of the network that can produce the highest amount of output with the fewest links. If I assume that forming and maintaining connections incurs an ex-ante fixed cost, and define the “overall efficiency” as the team’s output net of the total cost of forming and maintaining connections, then the network structure that achieves the highest overall efficiency depends on the relative magnitude of the fixed cost \( c_{ij}^F \). Here is the formal definition of overall efficiency. Note that the notion of overall efficiency is well-defined given the weak uniqueness of the equilibrium, since payoff profiles are the same across different equilibria.

**Definition 2.** The overall efficiency of a network \( G \), or \( OE(G) \), is the value

\[
OE(G) = f(e_1^*, ..., e_N^*) - \sum_{i,j \in A} g_{ij} \times c_{ij}^F,
\]

where \( e_i^* (i, j \in A = \{1, 2, ..., N\}) \) is the (uniquely determined) own effort in equilibrium.

If \( c_{ij}^F \) is small enough, then the network structure that achieves the highest overall efficiency is exactly the pairwise connected network as in the above corollary. If \( c_{ij}^F \) is large enough (approaching infinity), then it is possible that the network structure that achieves the highest overall efficiency is an empty network. Finally, if \( c_{ij}^F \) takes an intermediate value, then the network structure that achieves the highest overall efficiency is in between the pairwise connected and the empty network. To find such a network, one should calculate the output of each network that has fewer links than a pairwise connected one and then calculate the overall efficiency.

To be more specific, denote the maximum output of a network of \( N \) agents with \( L \) links \( y^*(N, L) \). Such a maximum exists since the possibility of such a network structure is finite. According to Proposition \( 6 \), we have the following lemma.

\(^{21}\)Since the cost of link maintenance is fixed, it does not matter who pays the cost. If there were a team manager (who in fact is not included in the current model) who is responsible for constructing the team, it is equivalent to the case that (s)he pays the cost.
Lemma 2. \( y^* (N, L_1) \leq y^* (N, L_2) \) if and only if \( L_1 \leq L_2 \).

The above lemma is built on the result of weak monotonicity of adding connections to a network. Given Lemma 2, we have the formal corollary for the network structure that achieves the highest overall efficiency. For simplicity, we assume that \( c_{ij}^F = c^F \) for any \( i \) and \( j \).

Corollary 2. The network structure that achieves the highest overall efficiency is the one that has \( L \) connections and that achieves the maximum output \( y^* (N, L) \) if and only if \( y^* (N, L) + (L' - L)c^F \geq y^* (N, L') \) for any \( L' \neq L \). In addition, \( L \) should satisfy \( 0 \leq L \leq N/2 \) if \( N \) is even and \( 0 \leq L \leq (N - 3)/2 + 3 \) if \( N \geq 3 \) is odd.

Corollary 2 can be easily obtained using the definition of overall efficiency. In all, given varying fixed costs, the most efficient network ranges from an empty one to a pairwise connected one. However, no matter what value the fixed cost takes, the most efficient network cannot have more connections, or be denser than the pairwise connected one.

Finally, the above results have important implications for organizational design. If there were a principal or manager (who in fact is not included in the model), he could exploit the theoretical results in this paper to build a team with an optimal social network, in the sense that the overall efficiency of the network attains the highest level possible (from the results of Corollary 2).

5 Concluding Remarks

To conclude, this article provided the first analysis of how mutual help matters for efficiency in a networked organization. Agents work collectively to produce the team’s output and are connected through a fixed, undirected, and unweighted social network. They can choose their own effort and helping efforts devoted to their network neighbors. I studied the subgame perfect Nash equilibria of the resulting two-stage game and addressed two questions: (1) Who helps whom and how much? and (2) How does the pattern of mutual help and the team’s performance rely on the network structure? I established the existence and weak uniqueness of this equilibrium and found that adding connectedness does not necessarily increase the team’s performance, which is an implication of the interplay of the externality of mutual help. Thus, I characterized the network structure in which the highest level of efficiency is achieved with the fewest links.

22The result of the optimal organizational structure justifies the existence of couple jobs, which are quite common in the labor market. Couples are better able to help each other, and, thus, raise the productivity of the entire organization.
In this article, the insights shed light on the emergence of several forms of the organization of production, especially pairwise connections in farms and workshops. Such insights are based on the assumption throughout this article that all agents in the network are homogeneous. The case of heterogeneity is not considered.

Finally, in this paper, I am also simply comparing the efficiency (defined as the “overall efficiency”) of different network structures, while not delving too deep into the discussion of a potential manager who is responsible for designing the incentive scheme and constructing the social network of the team. In real-world examples, a manager of a basketball team can select team members with the same cultural background (hence having connections) to foster better cooperation and “chemistry reactions.” An owner of a farm can also hire several different couples who work together in a pairwise connected manner, and within each couple, mutual help is an effective mechanism at play to ease work discomfort. Future research can provide further theoretical and empirical analysis regarding the structure of social networks and performance, especially taking into account the incentives and objectives of the manager.

References


Electronic copy available at: https://ssrn.com/abstract=3687345

Appendix A  Proofs

Details of construction of the auxiliary matrix We take into account two examples of $N = 3$. First, consider a line of three agents. The agents, from the left to the right, are agents 1, 2, and 3. Thus, there are four unknown helping efforts, and denote $h = (h_{12}, h_{21}, h_{23}, h_{32})^T$. Then, $AM(G)$ is a $2 \times 4$ matrix, and its matrix structure is the following.

$$
\begin{pmatrix}
1 & 0 & 0 & -1 \\
0 & 1 & -1 & 0
\end{pmatrix}
$$

The structure of $AM(G)$ corresponds to the linear system $AM(G)h = 0$ to solve for $h$. The linear equations are: (1) $h_{21} - h_{23} = 0$, and (2) $h_{12} - h_{32} = 0$. $AM(G)$ has full rank ($rank(AM(G)) = 2$). Thus, according to Proposition 4 the equilibrium is unique.

Second, consider a circle of three agents, agents 1, 2, and 3. Thus, there are six unknown helping efforts, and denote them by $h = (h_{12}, h_{13}, h_{23}, h_{21}, h_{31}, h_{32})^T$. Then, $AM(G)$ is a $6 \times 6$ matrix, and its matrix structure is the following.

$$
\begin{pmatrix}
1 & 1 & -1 & -1 & 0 & 0 \\
0 & 0 & 1 & 1 & -1 & -1 \\
1 & 0 & 0 & -1 & -1 & 1 \\
1 & -1 & -1 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & -1 & -1 \\
0 & -1 & -1 & 1 & 1 & 0
\end{pmatrix}
$$

The structure of $AM(G)$ corresponds to the linear system $AM(G)h = 0$ to solve for $h$. The linear equations are: (1) $h_{12} + h_{13} = h_{21} + h_{23}$, (2) $h_{21} + h_{23} = h_{31} + h_{32}$, (3) $h_{12} + h_{13} =$
\( h_{31} + h_{32}, \ (4) h_{21} + h_{31} = h_{12} + h_{32}, \ (5) h_{12} + h_{32} = h_{23} + h_{13}, \) and \( (6) h_{21} + h_{31} = h_{23} + h_{13}. \) \( \text{AM}(G) \) does not have full rank \( (\text{rank}(\text{AM}(G)) = 4 < 6). \) Thus, according to Proposition 4, the equilibrium is not unique.

**Details of deriving the best-reply functions:** From equation (1), we take a first-order condition, and thus the power of \( e_i \) should be \( w_0 - 1 \), and the first term in equation (1) should be multiplied by \( w_0 \). The second term in equation (1) is linear in \( e_i \), and thus taking a first-order derivative yields \( (\theta + \sum_{j \in N(i)} h_{ji})^{-\beta} \). Therefore, we have the following equation:

\[
\bar{\alpha} w_0 e_1^{w_0} e_2^{w_0} \cdots e_i^{w_0-1} \cdots e_N^{w_0} - (\theta + \sum_{j \in N(i)} h_{ji})^{-\beta} = 0. \tag{A3}
\]

Rearrange equation (A3) by expressing \( e_i \) as a function of \( e_j^* (j \neq i) \), and expressing \( e_j \) as a function of \( e_k^* (k \neq j) \) yield

\[
e_i = ((\theta + \sum_{j \in N(i)} h_{ji})^{-\beta} \prod_{j \neq i} (e_j^*)^{w_0}) ; e_j = ((\theta + \sum_{k \in N(j)} h_{kj})^{-\beta} \prod_{j \neq i} (e_j^*)^{w_0}) \tag{A4}
\]

Since equation (A4) holds for all \( i = 1, 2, ..., N \), comparing the equation for \( i \) and \( j \) yields (the common part \( \prod_{j \neq i} (e_j^*)^{w_0} \) cancels out)

\[
e_j^* = e_i \times \frac{(\theta + \sum_{k \in N(i)} h_{ki})^{-\beta}}{((\theta + \sum_{k \in N(j)} h_{kj})^{-\beta} \prod_{j \neq i} (e_j^*)^{w_0})}. \tag{A5}
\]

Express all \( e_j^* (j \neq i, j \in \{1, 2, ..., N\}) \) as a function of \( e_i \) by equation (A5), and plug back to equation (A4) (so that it is an equation regarding \( e_i \) only), we have

\[
e_i = (\alpha \tilde{w}_0) \times e_i^{w_0 N} \times ((\theta + \sum_{j \in N(i)} h_{ji})^\theta) \prod_{j \neq i} ((\theta + \sum_{k \in N(i)} h_{ki})^{-\beta w_0} N_j) \prod_{j=1}^N ((\theta + \sum_{k \in N(j)} h_{kj})^\beta w_0)). \tag{A6}
\]

Rearranging equation (A6) yields equation (2). Thus, we can express the payoff function of all helping efforts \( h_{ij} \), which is the right-hand side of the equation (3). Taking a first-order derivative of the first term of the equation (3) yields the left-hand side of the inequality (4). Taking a first-order derivative of the second term of the equation (3) yields the right-hand side of the inequality (4). Since \( h_{ij} \) should be weakly positive, the direction of the inequality (4)
should be “less than or equal to.” Standard complementary slackness conditions follow.

**Proof of Proposition 1** This proposition derives directly from Theorem 1 in Rosen (1965), which states that in an N-person concave game in which the strategy space of each agent is convex and compact, and the payoff function is concave in the agent’s own strategy, there always exists a Nash equilibrium. To prove that there exists a subgame perfect equilibrium of the subgame of stages 1 and 2 requires showing that the existence condition stated in Theorem 1 in Rosen (1965) is satisfied in the subgames of stage 2 and stage 1.

In stage 2, each agent i decides his or her level of own effort $e_i^*$ to maximize their payoff, taking the choices of other agents $\{e_j^*\}_{j \neq i}$ as given. Thus, the agent solves

$$e_i^* = \text{arg max}_{e_i} \bar{\alpha}e_1^{u_0}e_2^{u_0} \cdots e_i^{u_0} \cdots e_N^{u_0} - (\theta + \sum_{j \in N(i)} h_{ij})^{-\beta}e_i - \frac{1}{2}(\sum_{j \in N(i)} h_{ij})^2. \quad (A7)$$

If $e_i$ approaches $+\infty$, then the above equality cannot be satisfied. Thus, an upper limit can be imposed on the range of $e_i$. Then, it can be observed that the strategy space for $e_i$ is convex and compact and that the objective function is concave in $e_i$; therefore, there exists a Nash equilibrium in this subgame. The profile of $E^* = (e_1^*, \ldots, e_N^*)$ is given by equation (2).

By backward induction, substituting equation (2) into the payoff function, in stage 1, agent i’s problem is to find $\{h_{ij}\}_{j \in N(i)}$ given $\{h_{kl}^*\}_{k \neq i, l \in N(k)}$:

$$h_{ij}^* = \text{arg max}_{h_{ij}} \bar{\alpha}w_0 \frac{1}{1-w_0} \left( \prod_{k=1}^{N} (\theta + \sum_{l \in N(k)} h_{kl}^\beta)^{-\frac{w_0}{w_0}} - \gamma(\sum_{j \in N(i)} h_{ij})^2 \right). \quad (A8)$$

First, if $h_{ij}$ goes to $+\infty$, then the complementary slackness condition (4) may fail. So the solution $H^* = (h_{ij}^*)$ is bounded above. Again, the strategy space for $h_{ij}$ is convex and compact, and the objective function is concave in $h_{ij}$, given $\beta \in (0, \frac{1-Nw_0}{w_0})$. Therefore, there exists a Nash equilibrium of the subgame in stage 1, and there exists a subgame perfect Nash equilibrium in the overall game.

**Proof of Proposition 2(1):** Prove by contradiction. The first step is to show that every non-isolated agent gives a strictly positive amount of help. For such an agent $i$, let $j \in N(i)$
denote one of the agent’s neighbors. If, in equilibrium, the total amount of help \( i \) gives is zero, then the first-order condition for \( h_{ij} \) implies that

\[
0 < \frac{\beta w_0}{1 - N w_0} \prod_{j \geq 1} (\theta + \sum_{k \in N(j)} h_{kj}^*) \frac{\beta w_0}{1 - N w_0} (\theta + \sum_{k \in N(j)} h_{kj}^*) \left( \alpha w_0 \right)^{-1} (\bar{\alpha} w_0)^{-1} \left( \frac{1}{w_0} - 1 \right) \leq \gamma \left( \sum_{j \in N(i)} h_{ij}^* \right) = 0,
\]

and, hence, renders a contradiction.

The next step is to show that every non-isolated agent receives a strictly positive amount of help. For non-isolated agent \( i \), assume that all of his or her neighbors do not give him or her any help, that is, for any \( j \in N(i) \), \( h_{ji}^* = 0 \). There are two cases. (1) If one of \( i \)'s neighbors say \( j \), has no other neighbors except \( i \), then because all non-isolated agents give a strictly positive amount of help, we have \( h_{ji}^* > 0 \), which is a contradiction. (2) If all of \( i \)'s neighbors have other neighbors, except \( i \), i.e. for any \( j \in N(i) \), \( N(j) \) has no fewer than two elements, then because all non-isolated agents give a strictly positive amount of help, there exists \( k \in N(j)(k \neq i) \), such that \( h_{jk}^* > 0 \). Then, the first-order conditions of \( h_{ji} \) and \( h_{jk} \) imply that

\[
\frac{\beta w_0}{1 - N w_0} \prod_{j \geq 1} (\theta + \sum_{k \in N(j)} h_{kj}^*) \frac{\beta w_0}{1 - N w_0} (\theta + \sum_{k \in N(j)} h_{kj}^*) \left( \alpha w_0 \right)^{-1} (\bar{\alpha} w_0)^{-1} \left( \frac{1}{w_0} - 1 \right) \leq \gamma \left( \sum_{k \in N(j)} h_{jk}^* \right), \tag{A9}
\]

and

\[
\frac{\beta w_0}{1 - N w_0} \prod_{j \geq 1} (\theta + \sum_{k \in N(j)} h_{kj}^*) \frac{\beta w_0}{1 - N w_0} (\theta + \sum_{k \in N(j)} h_{kj}^*) \left( \alpha w_0 \right)^{-1} (\bar{\alpha} w_0)^{-1} \left( \frac{1}{w_0} - 1 \right) = \gamma \left( \sum_{k \in N(j)} h_{jk}^* \right). \tag{A10}
\]

As \( h_{jk}^* > 0 \), \( \theta < \theta + \sum_{k \in N(j)} h_{jk}^* \). Hence, comparing (A9) and (A10) \( \sum_{k \in N(j)} h_{jk}^* > \sum_{k \in N(j)} h_{jk}^* \), and, hence, a contradiction.

**Proof of Proposition 2(2):** The proof is straightforward, noting that for any \( i \) and \( j \), \((\sum_{j \in N(i)} h_{ij}^*)(\theta + \sum_{k \in N(j)} h_{kj}^*) \geq M(\mathbf{H}^*)\). \( h_{ij}^* > 0 \) implies that the first-order condition takes an equality sign, and, thus, for any \( j' \in N(i)/\{j\} \), \((\sum_{j \in N(i)} h_{ij}^*)(\theta + \sum_{k \in N(j')} h_{kj'}^*) \geq (\sum_{j \in N(i)} h_{ij}^*)(\theta + \sum_{k \in N(j)} h_{kj}^*)\). As \( \sum_{j \in N(i)} h_{ij}^* > 0 \), \( \sum_{k \in N(j)} h_{kj}^* \leq \sum_{k \in N(j)} h_{kj}^* \) for any \( j' \in N(i)/\{j\} \). By the same reasoning, if \( h_{ij}^* > 0 \), for any \( i' \in N(j)/\{i\} \), \( \sum_{j \in N(i)} h_{ij}^* \leq \sum_{j \in N(i')} h_{ij}^* \).
Proof of Proposition 2(3): This is a direct implication of the first-order condition (4).

For \( j_1, j_2 \in N(i) \) such that \( h^*_{ij_1}, h^*_{ij_2} > 0 \), the following two equations are obtained:

\[
\frac{\beta w_0}{1 - N w_0} \prod_{j \geq 1} \left( \theta + \sum_{k \in N(j)} h^*_{kj} \right)^{\frac{\theta w_0}{1 - N w_0}} \left( \theta + \sum_{k \in N(j)} h^*_{kj} \right)^{-1} \left( \bar{\alpha} w_0 \right)^{\frac{1}{1 - N w_0}} \left( \frac{1}{w_0} - 1 \right) = \gamma \left( \sum_{k \in N(j)} h^*_{jk} \right), (A11)
\]

\[
\frac{\beta w_0}{1 - N w_0} \prod_{j \geq 1} \left( \theta + \sum_{k \in N(j)} h^*_{kj_1} \right)^{\frac{\theta w_0}{1 - N w_0}} \left( \theta + \sum_{k \in N(j)} h^*_{kj_2} \right)^{-1} \left( \bar{\alpha} w_0 \right)^{\frac{1}{1 - N w_0}} \left( \frac{1}{w_0} - 1 \right) = \gamma \left( \sum_{k \in N(j)} h^*_{jk} \right), (A12)
\]

Comparing equations (A11) and (A12), \( \theta + \sum_{k \in N(j_1)} h^*_{kj_1} = \theta + \sum_{k \in N(j_2)} h^*_{kj_2} \), and, hence, \( j_1, j_2 \) receive the same amount of help. The second part of this proposition can be proved in the same way.

Proof of Proposition 3 and Corollary 1: Suppose that \( H^* \) has \( n \) agents and \( l \) links, where \( n \geq 2 \), and each node has at least degree one (thus, the network is a connected one). Thus, \( n - 1 \leq l \leq \frac{n(n-1)}{2} \). There are \( 2l \) unknown helping efforts (each link is associated with two helping efforts) in total. The sum of the degrees is also \( 2l \). As all the elements in \( H^* \) take interior solutions, Proposition 2(3) can be used to determine which helping efforts are equal, and this may yield \( 2(2l - n) = 4l - 2n \) equations.\(^{A1}\) Observe that the number of nodes \( n \) and the number of edges \( l \) satisfy the following inequality: \( n - 1 \leq l \leq \frac{n(n-1)}{2} \), and, thus, \( 2l - 2 \leq 4l - 2n \leq 4l - 2\sqrt{2l} \). This implies that there are, at most, two unknowns remaining. For a bipartite network, there are two independent sets, and each node in the same independent set gives and receives the same amount of helping effort, respectively, and each amount of given (received) helping effort is associated with one unknown. For a non-bipartite network, it then must be the case that all nodes receive and give the same amount of helping effort, as a result of Proposition 2(3).\(^{A1}\)

\(^{A1}\) With \( n \) nodes and \( l \) links, there are \( 2l - n \) pairs of neighbors, and, thus, \( 2(2l - n) = 4l - 2n \) equations for total helping efforts.
Proof of Proposition 4. The proof can be completed using a special version of the fixed-point theorem, proposed in [Kennan (2001)]. Without exploiting the fixed-point theorem, I can also prove this proposition, using the specific payoff structure of the game, as is illustrated below. The proof makes use of the monotonicity of the marginal benefit or cost functions or the concavity of the payoff function. By Proposition 3 if each $h_{ij}$ ($g_{ij} = 1$) in the profile of helping efforts $H$ takes an interior solution, then there are, at most, two unknowns, and there are, at most, two amounts of helping effort given and received, namely, $(H^g_1, H^g_2)$ and $(H^r_1, H^r_2)$. If there are exactly two unknowns, then $H^g_i$ and $H^r_i$ can be expressed as a (strictly increasing) linear transformation of the two unknowns (namely, $\tilde{h}_1$ and $\tilde{h}_2$), and, thus, $H^r_i = H^r_i(\tilde{h}_1, \tilde{h}_2)$, and $H^g_i = H^g_i(\tilde{h}_1, \tilde{h}_2)$, where $i \in \{1, 2\}$. The original set of first-order conditions is reduced to two and can be rewritten as

\[ M(H^r)(\theta + H^r_1(\tilde{h}_1, \tilde{h}_2))^{-1} = H^g_1(\tilde{h}_1, \tilde{h}_2), \]  

(A13) and

\[ M(H^r)(\theta + H^r_2(\tilde{h}_1, \tilde{h}_2))^{-1} = H^g_2(\tilde{h}_1, \tilde{h}_2). \]  

(A14)

The system of first-order conditions has only two degrees of freedom: the LHS is strictly decreasing in $(\tilde{h}_1, \tilde{h}_2)$, and the RHS is strictly increasing in $(\tilde{h}_1, \tilde{h}_2)$, and at least one solution exists by Proposition 1. Then, applying the two-dimensional Rolle’s theorem ([Furi and Martelli; 1995]) and exploiting the monotonicity of the LHS and RHS \(^A^2\) $(\tilde{h}_1, \tilde{h}_2)$ and, hence, $H^r_1$ and $H^g_2$ ($i \in \{1, 2\}$) can be uniquely determined.

If there is exactly one unknown, then there is one amount of helping effort given and received, namely, $H^g$ and $H^r$. Then, $H^g$ and $H^r$ can be expressed as a (strictly increasing) linear transformation of the only unknown (namely, $\tilde{h}$), and, thus, $H^r = H^r(\tilde{h})$, and $H^g = H^g(\tilde{h})$. There is only one distinct first-order condition:

\(^A^2\)By Rolle’s theorem, the system of equations (A13) and (A14) has at least one solution; by strict monotonicity, the solution is unique.
\[ M(H^*) - H^g(\hat{h})(\theta + H^r(\hat{h})) = 0. \] (A15)

Again, because the first-order condition has only one degree of freedom, and by the same reasoning as above, \( \hat{h} \) and, hence, \( H^r \) and \( H^g \) can be uniquely determined.

However, if some \( h_{ij} \) \((g_{ij} = 1)\) in the profile of helping efforts \( H \) take a corner solution, then, in equation (4), equality does not hold, and instead \( h_{ij} \) is directly pinned down by \( h_{ij} = 0 \). Therefore, the total degrees of freedom for \( H \) are the same when some \( h_{ij} \) take a corner solution, as in the case in which all \( h_{ij} \) take interior solutions.

Proof of Lemma 1: Prove by induction. The proof can be easily established for a circle of \( N = 3 \) agents (see equation (A2)). Suppose that the proof has been established for a circle of \( N = k \) agents. In a circle of \( N = k + 1 \) agents, the rank of \( AM(G) \) increases at most 1, but the dimension of \( AM(G) \) increases by 2. Thus, \( AM(G) \) must not be fully ranked for a circle of \( N = k + 1 \) agents.

Proof of Proposition 5: Recall that \( AM(G) \) is a \( 2K \times M \) matrix, where \( K \) is the number of rows of \( G \) with more than one non-zero entries and \( M \) is the number of unknown helping efforts. Thus, according to Proposition 2, the linear system to solve for unknown helping efforts is \( AM(G)h = 0 \). If \( AM(G) \) does not have full rank, i.e., \( rank(AM(G)) < \min\{2K, M\} \), then, based on Lemma 1, the graph \( G \) must contain a loop of at least three agents that leads to such a rank reduction (less than full rank). Using the proof process of Proposition 4, the number of equality restrictions is less than the number of unknown helping efforts. Since at least one solution exists (Proposition 1), the solution is not unique. Moreover, for a network without any loop (i.e., a tree), its leaf nodes, whose degrees equal 1, can be found. For these leaf nodes, the total amount of helping effort (both given and received) equals the helping effort on the single link between the nodes and their neighbor (For a leaf node \( i \) and its neighbor \( j \), then \( h^*_{ij} \) and \( h^*_{ji} \) are pinned down by \( M(H^*) = h^*_{ij}(\theta + \sum_{k \in N(j)} h^*_{kj}) \), and \( M(H^*) = \sum_{k \in N(j)} h^*_{jk}(\theta + h^*_{ji}) \), as a function of \( M(H^*) \), \( \sum_{k \in N(j)} h^*_{kj} \), and \( \sum_{k \in N(j)} h^*_{jk} \). Given all \( h^*_{ij} \) and \( h^*_{ji} \) of the leaf nodes, the helping efforts for all nodes with degree two can be pinned down, with the same method.
Given these helping efforts, the above reasoning for the rest of the nodes can be applied, until all helping efforts are substituted into equation (4) and are determined. Such is the unique way in which \( H^* \) is determined. For any network with loops, the proof is obvious because it can be derived directly from the example of a circle with \( N = 3 \), as shown in Figure 1.

**Proof of Proposition 6**: Suppose that in graph \( G \), a new link is added between nodes \( i \) and \( j \). Denote the equilibrium profile of helping efforts before the link is added by \( H^*_i \), and that, after the link is added by \( H^*_i \). The helping efforts between nodes \( i \) and \( j \) weakly increase after the link is added: \( h_{ij,2}^* \geq h_{ij,1}^* \) and \( h_{ji,2}^* \geq h_{ji,1}^* \), with the equality if the link added is a redundant link. The helping efforts given by \( i \) (\( j \)) to its neighbors also weakly increase, which is derived from the first-order condition, equation (4). Intuitively, these helping efforts and \( h_{ij}^* \) (\( h_{ji}^* \)) are strategic complements. However, the helping efforts given by \( i \)'s (\( j \)'s) neighbors to \( i \) (\( j \)) weakly decrease, which is also derived from the first order condition, equation (4). These helping efforts and \( h_{ij}^* \) (\( h_{ji}^* \)) are strategic substitutes.

So, the question is whether the effects of the decrease of helping efforts given by \( i \)'s (\( j \)'s) neighbors to \( i \) (\( j \)), and the subsequent reduction of the team’s output, will exceed the effects of the increase of the helping efforts between nodes \( i \) and \( j \) and the increase of the helping efforts given by \( i \) (\( j \)) to their neighbors, and thus the subsequent boost to the team’s output. This is possible only when the costs of the helping efforts of \( i \)'s (\( j \)'s) neighbors are reduced more than the reduction of the part of the team’s output they get, and, thus, their payoffs rise. However, if this is the case, then the payoffs of nodes \( i \) and \( j \) decrease, because the cost of helping efforts increases and the part of the team’s output they get decreases. Optimizing agents will not let this happen. Instead, they will set \( h_{ij,2}^* \) and \( h_{ji,2}^* \) to zero, so that their payoffs are unchanged.

However, adding a link does not necessarily increase the team’s output. There are cases in which the link added is a redundant one. The existence of redundant links is observed in the example in Figure B3(d), compared with Figure B3(c).

**Proof of Proposition 7**: By Proposition 3, the team’s performance reaches its maximum when the network \( G \) is complete, that is, every pair of nodes are linked. In this case, each node gives and receives the same amount of helping effort, and the equilibrium helping effort is given by

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\[
\gamma^{-1} \frac{\beta w_0}{1 - N w_0} \prod_{j \geq 1} (\theta + \tilde{h})^{\frac{\beta w_0}{\alpha w_0}} (\bar{\alpha} w_0)^{\frac{1}{N w_0}} \left( \frac{1}{w_0} - 1 \right) - \tilde{h}(\theta \tilde{h}) = 0, \tag{A16}
\]

where \( \tilde{h} \) is the total helping effort given and received by each node. When the network has even nodes and is pairwise linked (all the nodes are linked in pairs), then the total amount of helping effort given and received is also equal across nodes and is also pinned down by the above first-order condition (A16). This is also the case when the network has an odd number of nodes, with \( N - 3 \) nodes pairwise linked and the remaining three nodes linked in a circle.

Finally, in the pairwise linked network, all links are non-redundant, so deleting any link will cause the helping efforts and, hence, the team’s output, to decrease. In addition, if one or more links in the network are moved elsewhere, and form a different network, there will be no way to achieve the equilibrium described by (A16). Therefore, the pairwise linked network is the one that can attain the highest level of performance with the fewest links.

Proof of Lemma 2: The proof follows from Proposition 3, which states that the output of the network weakly increases as new connections are added. Thus, \( y^*(N, L_1) \leq \tilde{y} \leq y^*(N, L_2) \), where \( \tilde{y} \) is the output of the network which is constructed by adding \( L_2 - L_1 \) links on the network that has \( L_1 \) links and that can maximize the output among all networks that have \( L_1 \) links.

Proof of Corollary 2: The proof follows directly from Lemma 2 and the definition of overall efficiency. In addition, the network structure that is the most overall efficient is the one with links fewer than the pairwise connected network.

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Appendix B  Examples

B.1 Cases with $N = 3$

Case with $N = 2$  Consider the case in which there are two agents who are connected to each other, which is shown in Figure B1(a). By Proposition [2](1), agents 1 and 2 both provide a strictly positive amount of helping effort to each other, that is, $h_{12}^*, h_{21}^* > 0$. Hence $h_{12}^*, h_{21}^*$ are the solution to the following equations:

\[
\frac{\beta w_0}{1 - 2w_0} \left( \theta + h_{12}^* \right) \frac{\delta w_0}{1 - 2w_0}^{-1} \left( \theta + h_{12}^* \right) \frac{\alpha}{2w_0} \left( \frac{1}{w_0} - 1 \right) = \gamma h_{12}^*, \tag{B1}
\]

\[
\frac{\beta w_0}{1 - 2w_0} \left( \theta + h_{12}^* \right) \frac{\delta w_0}{1 - 2w_0}^{-1} \left( \theta + h_{12}^* \right) \frac{\alpha}{2w_0} \left( \frac{1}{w_0} - 1 \right) = \gamma h_{21}^*, \tag{B2}
\]

Comparing these two equations, $(\theta + h_{12}^*)h_{12}^* = (\theta + h_{21}^*)h_{21}^*$, and, hence, $h_{12}^* = h_{21}^* = \tilde{h}$. Then, substituting into either of the two equations yields

\[
\frac{\beta w_0}{1 - 2w_0} \left( \theta + \tilde{h} \right) \frac{\delta w_0}{1 - 2w_0}^{-1} \left( \theta + h_{12}^* \right) \frac{\alpha}{2w_0} \left( \frac{1}{w_0} - 1 \right) = \gamma \tilde{h}. \tag{B3}
\]

By Rolle’s theorem, and exploiting the monotonicity of the LHS and RHS this equation has a unique solution. Hence, the equilibrium is unique and symmetric.

First, consider a star network with $N = 3$, shown in Figure B1(b). Again, by Proposition [2](1), $h_{12}^*, h_{21}^*, h_{23}^*, h_{32}^* > 0$. By Proposition [2](3), $h_{12}^* = h_{32}^* \equiv h_i^*$, $h_{21}^* = h_{23}^* \equiv h_i^*$. Hence, $h_1^*, h_2^*$

Figure B1: Cases of $N \leq 3$

First, consider a star network with $N = 3$, shown in Figure B1(b). Again, by Proposition [2](1), $h_{12}^*, h_{21}^*, h_{23}^*, h_{32}^* > 0$. By Proposition [2](3), $h_{12}^* = h_{32}^* \equiv h_i^*$, $h_{21}^* = h_{23}^* \equiv h_i^*$. Hence, $h_1^*, h_2^*$
are the solution to the following equations

$$\begin{align*}
\frac{\beta w_0}{1 - 3w_0} (\theta + 2h_1^*)^{2} & (\theta + h_2^*)^{\frac{\beta w_0}{3w_0} - 1} (\theta + h_3^*)^{\frac{\beta w_0}{3w_0} - 1} (\frac{\alpha}{3} w_0) \frac{1}{w_0} (\frac{1}{w_0} - 1) = \gamma h_1^*, \\
\frac{\beta w_0}{1 - 3w_0} (\theta + 2h_1^*)^{2} (\theta + h_2^*)^{\frac{\beta w_0}{3w_0} - 1} (\theta + h_3^*)^{\frac{\beta w_0}{3w_0} - 1} (\frac{\alpha}{3} w_0) \frac{1}{w_0} (\frac{1}{w_0} - 1) = 2\gamma h_2^*.
\end{align*}$$

(B4) (B5)

Similarly, these equations have a unique solution \(h_1^*, h_2^*\). Simple algebra shows that \(h_1^* < h_2^* < 2h_1^*\), and, thus, the central agent gives and receives more help than the peripheral agents.

The agents’ own equilibrium effort \(E^*\) is then given by substituting \(H^*\) into equation (2).

Then consider a circle of three agents, as shown in Figure B1c. As by Proposition 2 (1), each agent gives and receives a strictly positive amount of help, it can be assumed that \(h_{12}^*, h_{23}^*, h_{31}^* > 0\). If none of \(h_{12}^*, h_{23}^*, h_{31}^*\) takes the interior solution, then the first-order conditions \((4)\) for \(h_{21}^*, h_{32}^*, h_{13}^*\) all take strictly less than equal sign. Then, \(h_{12}^* > h_{23}^* > h_{31}^* > h_{12}^*\), which is a contradiction. If one of \(h_{21}^*, h_{32}^*, h_{13}^*\) takes the interior solution, assume it is \(h_{21}^*\), then by Proposition 2 (3) we have \(h_{12}^* + h_{31}^* = h_{23}^*\). Hence, \(h_{12}^* \leq h_{23}^* < h_{12}^*\), which is a contradiction. Similarly, it can be shown that if two of \(h_{21}^*, h_{32}^*, h_{13}^*\) take the interior solution there will also be a contradiction. Therefore, all helping efforts take interior solutions, and by Proposition 2 (3), all agents give and receive an identical amount of help, \(\tilde{h}_{B2}\) which is determined by

$$\begin{align*}
\frac{\beta w_0}{1 - 3w_0} (\theta + \tilde{h})^{\frac{3\beta w_0}{3w_0} - 1} (\frac{\alpha}{3} w_0) \frac{1}{w_0} (\frac{1}{w_0} - 1) = \gamma \tilde{h}.
\end{align*}$$

(B6)

As \(\tilde{h}\) is uniquely determined, the agents’ own equilibrium effort \(E^*\) is then uniquely given by substituting \(H^*\) into equation (2). However, note that the equilibrium \(H^*\) is not unique. Even though the total helping efforts given and received are the same, who gives help to and receives help from whom may not be unique. The three examples in Figure 1 illustrate this point. The helping efforts in the same directions as the arrows take an interior solution.

Redundant Links Where the Associated Helping Efforts Take a Corner Solution

In a network in which there are nodes with high degrees (more than two), the links among these nodes are redundant links. The following is an example of an equilibrium with the existence of

\(^{B2}\)The total amounts of help received and given are equal.
a redundant link:

**Example B1.** *(A case with a redundant link)* Consider the network in Figure B3(a). The link between agent 2 and agent 3 is a redundant link.

![Figure B2: Types of agents](image)

The detailed discussion of this example, and the following examples, is relegated to the Appendix C.

**B.2 Cases with \( N = 4 \)**

**Cases with \( N = 4 \)** First, consider a line with four agents, as shown in Figure B3(a).

By Proposition 2(1), \( h_{12}^*, h_{21}^*, h_{43}^*, h_{34}^* > 0 \). Then it can be shown that \( h_{23}^* \) and \( h_{32}^* \) take interior solutions. If only one of \( h_{23}^* \) and \( h_{32}^* \) takes an interior solution, then assume that it is \( h_{23}^* \), and hence, \( h_{23}^* + h_{21}^* = h_{43}^* \), and \( h_{21}^* = h_{23}^* + h_{43}^* \). Thus, \( h_{23}^* = 0, h_{21}^* = h_{43}^* \). First-order conditions imply that \( (\theta + h_{12}^*)h_{12}^* = (\theta + h_{13}^*)h_{13}^* = (\theta + h_{43}^*)h_{43}^* = (\theta + h_{34}^*)h_{34}^* \). Hence, \( h_{12}^* = h_{21}^* = h_{43}^* = h_{34}^*; h_{23}^* = h_{32}^* = 0 \); and both \( h_{23}^* \) and \( h_{32}^* \) take interior solutions, which is a contradiction. Similarly, it can be shown that it is also infeasible if neither \( h_{23}^* \) nor \( h_{32}^* \) takes an interior solution. Therefore, the equilibrium helping efforts are \( h_{12}^* = h_{21}^* = h_{43}^* = h_{34}^* = \tilde{h} \), \( h_{23}^* = h_{32}^* = 0 \), where \( \tilde{h} \) is determined uniquely by

\[
\frac{\beta w_0}{1 - 4 w_0} (\theta + \tilde{h}) \frac{4 w_0}{1 - 4 w_0} - 1 \left( \frac{\alpha}{4 w_0} \right) \frac{1}{1 - 4 w_0} (\frac{1}{w_0} - 1) = \gamma \tilde{h}. \tag{B7}
\]

The connection between agents 2 and 3 is redundant: deleting this link would also result in the interior solution being zero.

\[B3\] Here the interior solution is the zero solution.
this equilibrium, as in Figure B3(b). The principal can thus avoid the fixed cost of maintaining
the connection between agents 2 and 3 by cutting their connection.

Next, consider a circle of four agents, as in Figure B3(c). The difference, compared with
the former example, is that an extra link is added between agents 1 and 4. By the same
reasoning as the case of the circle of three agents, all agents give and receive the same amount
of helping effort $\hat{h}$, which is uniquely determined by equation (B7). The specific pattern of $H^*$,
the same as in the case of $n = 3$, is not unique. Hence, this is an example in which adding a
new link between two agents does not increase their own effort and the team’s output. Also,
it can be shown that in a complete network of four agents, as in Figure B3(d), the equilibrium
helping effort given by and received from each agent is the same for the line and the circle. The
following subsection shows that this observation applies to any network structure with an even
number of agents.

Figure B3: Examples with four agents

Figure B3(a) and (d) show the examples where there are redundant links associated with
the helping efforts being zero but that take interior solutions. The following is an example of a
network that has a redundant link associated with helping efforts that take a corner solution.

B.3 Generalization of $N$ Agents

Star of $N$ Agents  Consider a star with $N$ agents, in which there is a central node and
$N - 1$ peripheral nodes. Let the central agent be agent 1 and $\{2, 3, ..., N\}$ be the peripheral
nodes.
Example B2. (Equilibrium of a Star) In a star with $N$ agents, there is a unique equilibrium $H^*$ and $E^*$ such that each peripheral agent gives to and receives from the central agent the same amount of helping effort: for all $i \in \{2, 3, ..., N\}$, $h^*_{i1} \equiv \tilde{h}_1 > 0, h^*_{1i} \equiv \tilde{h}_2 > 0$, where $\tilde{h}_1 > 0, \tilde{h}_2 > 0$ are uniquely determined by equations (B8) and (B9).

Then, by Proposition 2(1) and (3), for all $i \in \{2, 3, ..., N\}$, $h^*_{i1} = \tilde{h}_1 > 0$, $h^*_{1i} = \tilde{h}_2 > 0$, where $\tilde{h}_1 > 0, \tilde{h}_2 > 0$ are uniquely determined by

$$\beta w_0 \left[\theta + (N - 1)\tilde{h}_1\right]^{\frac{\beta w_0}{1 - N w_0} - 1} \left(\theta + \tilde{h}_2\right) \frac{(N - 1)\beta w_0}{1 - N w_0} \left(\tilde{h}_1\right) \frac{1}{w_0} - 1 = \gamma \tilde{h}_1, \quad (B8)$$

$$\beta w_0 \left[\theta + (N - 1)\tilde{h}_1\right]^{\frac{\beta w_0}{1 - N w_0} - 1} \left(\theta + \tilde{h}_2\right) \frac{(N - 1)\beta w_0}{1 - N w_0} \left(\tilde{h}_2\right) \frac{1}{w_0} - 1 = \gamma \tilde{h}_2. \quad (B9)$$

By simple algebra, $[\theta + (N - 1)\tilde{h}_1] \tilde{h}_1 = (\theta + \tilde{h}_2)(N - 1)\tilde{h}_2$, and, hence, $\tilde{h}_1 < (N - 1)\tilde{h}_2$ and $\tilde{h}_2 < (N - 1)\tilde{h}_1$. The central node gives and receives more help.

**Line of $N$ Agents** Consider a line with $N$ agents. The equilibrium of the network structure is characterized as follows:

Example B3. (Equilibrium of a Line) In a line with $N$ agents, for any two neighboring agents $i$ and $i + 1$, the helping efforts $h^*_{i,i+1}$ and $h^*_{i+1,i}$ take interior solutions, and the equilibrium helping effort and own effort $H^*$ and $E^*$ are unique.

When $N$ is even, say $N = 2K$ ($K \in \mathbb{N}$), then by the examples of $N = 2$ and $N = 4$ and induction, all agents give and receive the same amount of help, $\tilde{h}$, which is determined by

$$\beta w_0 \left(\theta + \tilde{h}\right) ^{\frac{\beta w_0}{1 - Nw_0} - 1} \left(\tilde{h}_1\right) \frac{1}{w_0} - 1 = \gamma \tilde{h}. \quad (B10)$$

When $N$ is odd, say $N = 2K + 1$ ($K \in \mathbb{N}$), then there are two independent sets and correspondingly two types of agents. All agents of each type give and receive the same amount of help. Hence, by simple algebra:
\[
\frac{\beta w_0}{1 - N w_0} (\theta + (K + 1)\tilde{h}_1)^{\frac{K\beta w_0}{1 - N w_0} - 1} (\theta + K\tilde{h}_2)^{\frac{(K + 1)\beta w_0}{1 - N w_0} - 1} (\tilde{\alpha} w_0)^{\frac{1}{1 - N w_0} (\frac{1}{w_0} - 1)} = K\gamma \tilde{h}_1, \quad \text{(B11)}
\]

\[
\frac{\beta w_0}{1 - N w_0} (\theta + (K + 1)\tilde{h}_1)^{\frac{K\beta w_0}{1 - N w_0} - 1} (\theta + K\tilde{h}_2)^{\frac{(K + 1)\beta w_0}{1 - N w_0} - 1} (\tilde{\alpha} w_0)^{\frac{1}{1 - N w_0} (\frac{1}{w_0} - 1)} = (K + 1)\gamma \tilde{h}_2. \quad \text{(B12)}
\]

Nodes in the same independent set as the leaf node of the line give \(K\tilde{h}_1\) helping efforts and receive \(K\tilde{h}_2\). Nodes in the other independent set give \((K + 1)\tilde{h}_2\) helping efforts and receive \((K + 1)\tilde{h}_1\).

**Circle of \(N\) Agents**  Consider a circle with \(N\) agents. The equilibrium of the network structure is characterized as follows:

**Example B4. (Equilibrium of a Circle)** In a circle with \(N\) agents, for any two neighboring agents \(i\) and \(i + 1\), the helping effort \(h_{i,i+1}^*\) and \(h_{i+1,i}^*\) take interior solutions. In addition, the profile of the agent’s own effort \(E^*\) is unique, but the helping effort \(H^*\) is not.

In the circle of \(N\) agents, each agent gives and receives the same total amount of helping efforts \(\tilde{h}\), which is uniquely determined by

\[
\frac{\beta w_0}{1 - N w_0} (\theta + \tilde{h})^{\frac{N\beta w_0}{1 - N w_0} - 1} (\tilde{\alpha} w_0)^{\frac{1}{1 - N w_0} (\frac{1}{w_0} - 1)} = \gamma \tilde{h}. \quad \text{(B13)}
\]

However, as illustrated by the previous examples, the specific pattern of help is not unique.

The following is an example that characterizes equilibria in a general way:

**Example B5. (Generic bipartite graphs and non-bipartite graphs)** For a bipartite graph, the graph underlying the equilibrium helping efforts \(H^*_B\) may be divided into several bipartite subgraphs, and in each subgraph, the nodes in the same independent set give and receive the same total amount of help. For a non-bipartite graph, like the one in Figure B4, the graph underlying the equilibrium helping efforts \(H^*\) may be divided into several bipartite or non-bipartite subgraphs. For the graph in Figure B4, the entire graph can be cut into three bipartite subgraphs: (1) agent 4-5; (2) agent 1-2; and (3) agent 3-6. In each bipartite subgraph, the nodes

---

\(^{B4}\text{Recall that the network } G \text{ underlying } H^* \text{ is a graph in which } g_{ij} = 1 \text{ if and only if } h_{ij}^* > 0.\)
in the same independent set give and receive the same amount of help, and in each non-bipartite subgraph, all nodes give and receive the same amount of help.

Appendix C  Details of the Examples

Details of Example B1 Apply the steps described in the beginning of Section 4.3. First, assume that all helping efforts take an interior solution. This yields the following equalities:

\[ h^*_{12} = h^*_{32}, \ h^*_{21} = h^*_{23} + h^*_{43} + h^*_{63}, \ h^*_{53} = h^*_{43} = h^*_{63} = h^*_{23} + h^*_{21}, \ h^*_{34} = h^*_{35} = h^*_{36} = h^*_{32} + h^*_{12}. \]

From these equalities and \( h^*_{21}, h^*_{34}, h^*_{35}, h^*_{36} > 0 \) (Proposition 2(1)), it can be derived that: \( h^*_{23} \) and \( h^*_{32} \) must take a corner solution; otherwise, \( h^*_{21}, h^*_{34} = 0 \), which is a contradiction. Thus, the link between agents 2 and 3 is redundant. The remaining unknowns are \( h^*_{12} \equiv h_1, h^*_{21} \equiv h_2, h^*_{34} = h^*_{35} = h^*_{36} \equiv h_3, \) and \( h^*_{43} = h^*_{53} = h^*_{63} \equiv h_4 \). Then, the first-order conditions are as follows:

\[ M(H^*)(\theta + h_1)^{-1} = h_1, \ M(H^*)(\theta + h_2)^{-1} = h_2, \ M(H^*)(\theta + h_3)^{-1} = 3h_3, \]
\[ M(H^*)(\theta + h_4)^{-1} = h_4. \]

These four equalities pin down the equilibrium for this example. \( \square \)

Details of Example 1 For networks before adding the links in Figure 2(a) and (b), nodes 1, 3, and 5 belong to one independent set, and nodes 2 and 4 belong to the other. The profile of helping efforts before adding the link is:

\[ h^*_{12} = 2h^*_{32} = 2h^*_{34} = h^*_{35} \equiv 2h_1, \]
\[ h^*_{21} = 2h^*_{23} = 2h^*_{43} = h^*_{45} \equiv 2h_2. \]

\( \hat{h}_1 \) and \( \hat{h}_2 \) are pinned down by

\[ \frac{\beta w_0}{1 - 5w_0} (\theta + 3\hat{h}_1)^{2\beta w_0} (\theta + 2\hat{h}_2)^{3\beta w_0} \left( \frac{\alpha}{5} w_0 \right)^{\frac{\beta w_0}{1 - 5w_0}} \left( \frac{1}{w_0} - 1 \right) = \gamma 2\hat{h}_1, \]  
(C1)

\[ \frac{\beta w_0}{1 - 5w_0} (\theta + 3\hat{h}_1)^{2\beta w_0} (\theta + 2\hat{h}_2)^{3\beta w_0} \left( \frac{\alpha}{5} w_0 \right)^{\frac{\beta w_0}{1 - 5w_0}} \left( \frac{1}{w_0} - 1 \right) = 3\hat{h}_2. \]  
(C2)

For Figure 2(a), assume that the equilibrium profile of helping efforts after adding the link,
\( \mathbf{H}^\ast \), takes an interior solution for all \( h_{ij}^\ast \). Then, by Proposition 2(3), the following set of equalities is obtained:

\[
\begin{align*}
   & h_{12}^\ast + h_{14}^\ast = h_{32}^\ast + h_{54}^\ast, \\
   & h_{21}^\ast + h_{23}^\ast = h_{23}^\ast + h_{43}^\ast = h_{34}^\ast, \\
   & h_{21}^\ast + h_{23}^\ast = h_{23}^\ast + h_{34}^\ast + h_{15}^\ast,
\end{align*}
\]

and \( h_{12}^\ast + h_{32}^\ast = h_{14}^\ast + h_{34}^\ast + h_{54}^\ast \). Thus, there are two equilibria: (1) \( h_{12}^\ast = 2h_{14}^\ast = 2h_{32}^\ast = 2h_{54}^\ast \), \( h_{21}^\ast = 2h_{23}^\ast = 2h_{43}^\ast = 2h_{34}^\ast \), and \( h_{12}^\ast = 2h_{32}^\ast = 2h_{14}^\ast = 0 \); (2) \( h_{32}^\ast = 2h_{14}^\ast = 2h_{34}^\ast = h_{54}^\ast \), \( h_{23}^\ast = 2h_{21}^\ast = 2h_{41}^\ast = h_{45}^\ast \), and \( h_{34}^\ast = 2h_{43}^\ast = 0 \). These equations inform us that all helping efforts take an interior solution and that in either equilibrium, the team’s output is unchanged after adding the link.

For Figure 2(b), after adding the link, the network becomes a circle. Hence, apply the example of circles discussed above, all the nodes give and receive the same amount of helping effort \( \bar{h} \), which is determined by

\[
\frac{\beta w_0}{1 - 5w_0} (\theta + \bar{h}) \frac{5\beta w_0}{1 - 5w_0} \left( \frac{\alpha}{w_0} \right)^{\frac{1}{1 - 5w_0}} \left( \frac{1}{w_0} - 1 \right) = \gamma \bar{h}.
\]

(C3)

Thus, the equilibrium condition for the helping effort has changed, and, by Proposition 6, the team’s output strictly increases.

For Figure 2(c), before adding the link, nodes 1 and 3 belong to one independent set, and nodes 2 and 4 belong to the other. The equilibrium helping efforts are \( h_{12}^\ast = h_{21}^\ast = h_{34}^\ast = h_{54}^\ast = \bar{h}, h_{23}^\ast = h_{32}^\ast = 0 \), where \( \bar{h} \) is determined uniquely by

\[
\frac{\beta w_0}{1 - 4w_0} (\theta + \bar{h}) \frac{4\beta w_0}{1 - 4w_0} \left( \frac{\alpha}{w_0} \right)^{\frac{1}{1 - 4w_0}} \left( \frac{1}{w_0} - 1 \right) = \gamma \bar{h}.
\]

(C4)

Assume that the equilibrium profile of helping efforts after adding the link, \( \mathbf{H}^\ast \), takes the interior solution for all \( h_{ij}^\ast \). Then, by Proposition 2(3), the following set of equalities can be obtained:

\[
\begin{align*}
   & h_{12}^\ast + h_{13}^\ast = h_{21}^\ast + h_{23}^\ast = h_{31}^\ast + h_{32}^\ast + h_{34}^\ast = h_{43}^\ast, \\
   & h_{21}^\ast + h_{31}^\ast = h_{13}^\ast + h_{23}^\ast + h_{43}^\ast = h_{12}^\ast + h_{32}^\ast = h_{54}^\ast. 
\end{align*}
\]

Thus, \( h_{12}^\ast = h_{21}^\ast = h_{34}^\ast = h_{43}^\ast \), and \( h_{13}^\ast = h_{31}^\ast = h_{23}^\ast = h_{32}^\ast = 0 \). Adding the link does not change the team’s output.

Details of the Example of a Line: The proof consists of three steps. Step 1: prove that the equilibrium in which the helping effort between any two neighboring agents takes interior solutions exists and is unique. Step 2: prove that there are no neighboring agents \( i \)
and \(i + 1\) such that only one of \(h_{i,i+1}^*\) and \(h_{i+1,i}^*\) takes an interior solution. Step 3: prove that, in equilibrium, there are no neighboring agents \(i\) and \(i + 1\) such that neither \(h_{i,i+1}^*\) nor \(h_{i+1,i}^*\) takes interior solutions. These three steps show that the equilibrium obtained in step 1 is the unique equilibrium.

Step 1. There are two cases: when \(N\) is even and when \(N\) is odd. When \(N\) is even, say \(N = 2K\), then all agents give and receive the same amount of help, \(\tilde{h}\), which is determined by

\[
\frac{\beta w_0}{1 - N w_0} (\theta + \tilde{h}) \frac{1}{1 - N w_0} (\tilde{\alpha} w_0) \frac{1}{w_0} - 1 = \gamma \tilde{h}.
\]  

(C5)

When \(N\) is odd, say \(N = 2K + 1\), then by Proposition 2(3), there are two independent sets, or two types of agents. All agents in each type give and receive the same amount of help. Hence, by simple algebra,

\[
\frac{\beta w_0}{1 - N w_0} (\theta + (K + 1) \tilde{h}_1) \frac{1}{1 - N w_0} (\tilde{\alpha} w_0) \frac{1}{w_0} - 1 = K \gamma \tilde{h}_1,
\]  

(C6)

\[
\frac{\beta w_0}{1 - N w_0} (\theta + (K + 1) \tilde{h}_1) \frac{1}{1 - N w_0} (\tilde{\alpha} w_0) \frac{1}{w_0} - 1 = (K + 1) \gamma \tilde{h}_2.
\]  

(C7)

Step 2. Prove by contradiction. Assume that there are two neighboring agents \(i\) and \(i + 1\) such that only one of \(h_{i,i+1}^*\) and \(h_{i+1,i}^*\) takes an interior solution. Without loss of generality, assume that \(h_{i+1,i}^*\) takes a corner solution. By Proposition 2(1), agent \(i + 1\) has another neighbor \(i + 2\), and agent \(i\) has another neighbor \(i - 1\), and \(h_{i-1,i}^*\) and \(h_{i+1,i+2}^*\) take interior and strictly positive solutions. Hence, \(h_{i+1,i+2}^*(\theta + h_{i-1,i}^*) > (h_{i-1,i-2}^*)^*(\theta + h_{i-1,i}^*)\), and \(h_{i+1,i+2}^*(\theta + h_{i+1,i+2}^* + h_{i+3,i+2}^*) < h_{i+1,i+2}^*(\theta + h_{i-1,i}^*)\), which is a contradiction because all helping efforts are nonnegative.

Step 3. Prove by induction. Section 4.3.1 shows that it is true when \(N = 2\). Assume that it is true when \(N = K - 1(K > 2)\). When \(N = K\), if there exist \(i\) and \(i + 1\) such that both \(h_{i,i+1}^*\) and \(h_{i+1,i}^*\) take corner solutions, then, by Proposition 2(1), \(i\) and \(i + 1\) cannot be located at the two ends of the line. Hence, \(i\) and \(i + 1\) divide the entire link into two parts:
\( A_1 = \{1, 2, \ldots, i\} \) and \( A_2 = \{i + 1, i + 2, \ldots, K\} \). Each part of the entire line consists of fewer than \( K - 1 \) agents, and, hence, by assumption, helping efforts between any two neighboring agents take interior solutions. By step 1, the equilibrium helping effort received from and given by each agent is uniquely determined. The next step is to show that the assumption that both \( h_{i,i+1}^* \) and \( h_{i+1,i}^* \) take a corner solutions leads to a contradiction. As \( h_{i,i+1}^* \) takes a corner solution, \( h_{i,i+1}^* + h_{i+1,i+1}^* < h_{i+2,i+1}^* + h_{i+2,i+2}^* \), and \( h_{i+1,i+2}^* > h_{i-1,i}^* + h_{i-1,i-2}^* \). Thus, the contradicts the fact that all helping efforts are non-negative.

**Details of the Example of a Circle:** The proof consists of three steps, which is similar to the case of a line. Step 1: prove that the equilibrium in which helping efforts between any two neighboring agents take interior solutions exists, and that the profile of total helping efforts given and received is unique. Step 2: prove that there are no neighboring agents \( i \) and \( i + 1 \) such that only one of \( h_{i,i+1}^* \) and \( h_{i+1,i}^* \) takes an interior solution. Step 3: prove that there are no neighboring agents \( i \) and \( i + 1 \) such that neither \( h_{i,i+1}^* \) nor \( h_{i+1,i}^* \) takes interior solutions. These three steps show that the profile of total helping effort given and received, obtained in step 1, is unique in equilibrium. However, the specific pattern of helping effort is not unique, as was shown in the previous examples.

Step 1. In the circle of \( N \) agents, by Proposition 2(3), each agent gives and receives the same amount of effort \( \tilde{h} \), which is uniquely determined by

\[
\frac{\beta w_0}{1 - \frac{\bar{\alpha} w_0}{1}} (\theta + \tilde{h})^{\frac{N \bar{\alpha} w_0}{1}}(\tilde{h}^{\frac{1}{1-Nw_0}} (\frac{1}{w_0} - 1) = \gamma \tilde{h}. \tag{C8}
\]

However, as illustrated by the previous examples, the specific pattern of mutual help is not unique.

Step 2. Prove by contradiction. Assume that there are two neighboring agents \( i \) and \( i + 1 \) such that only one of \( h_{i,i+1}^* \) takes an interior solution. Suppose that \( h_{i+1,i}^* \) takes the corner solution. By Proposition 2(1), agent \( i + 1 \) has another neighbor \( i + 2 \), and agent \( i \) has another
Thus, $h_{i-1,i}$ and $h_{i+1,i+2}$ take interior solutions. Hence, $h_{i+1,i+2}(\theta + h_{i-1,i}^*) > (h_{i-1,i-2}^* + h_{i-1,i}^*)(\theta + h_{i-1,i}^*)$, and $h_{i+1,i+2}(\theta + h_{i+1,i+2}^* + h_{i+3,i+2}^*) < h_{i+1,i+2}^*(\theta + h_{i-1,i}^*)$, which is a contradiction because all helping efforts are nonnegative.

Step 3. Prove by contradiction. Assume that there exist $i$ and $i+1$ such that both $h_{i,i+1}$ and $h_{i+1,i}$ take corner solutions, then, the circle becomes a line whose two ends are $i$ and $i+1$. By Proposition 2(1), an end of a chain always gives and receives less help than the neighboring agent near the middle. Hence, the two inequalities $h_{i,i-1}^*(\theta + h_{i+1,i+2}^*) > M(H^*)$ and $h_{i+1,i+2}^*(\theta + h_{i-1,i}^*) > M(H^*)$ cannot hold at the same time, which is a contradiction.

Details of the Example of a Non-Bipartite Graph: Consider the non-bipartite graph in Figure B4. First, assume that all helping efforts take an interior solution. Then, by Proposition 2(3), the following set of equalities is obtained: $h_{12}^* + h_{13}^* + h_{14}^* = h_{31}^* + h_{32}^* + h_{36}^* = h_{54}^* = h_{21}^* + h_{23}^* = h_{63}^*$, and $h_{21}^* + h_{31}^* + h_{41}^* = h_{13}^* + h_{23}^* + h_{63}^* = h_{12}^* + h_{32}^* + h_{36}^* = h_{45}^* + h_{54}^*$. Thus, $h_{13}^* = h_{31}^* = h_{14}^* = h_{41}^* = h_{23}^* = h_{52}^* = 0$ (but all take an interior solution), and $h_{12}^* = h_{21}^* = h_{36}^* = h_{45}^* = h_{54}^* \equiv \hat{h} > 0$. $\hat{h}$ is pinned down by $\frac{\beta u_0}{1-\delta w_0} (\theta + \hat{h})^{\frac{\delta w_0}{\delta w_0} - 1} (\tilde{\alpha} w_0) \cdot \frac{1}{\tilde{\alpha} w_0} (\frac{1}{\tilde{\alpha} w_0} - 1) = \gamma \hat{h}$. Thus, in equilibrium, the graph underlying the helping efforts $H^*$ constitutes three bipartite subgraphs.

Appendix D  Extensions and Modeling Assumptions Revisited

In this subsection, I revisit the main assumptions, including the functional form assumptions, perfect substitutability of helping efforts, and homogeneity assumption. I also discuss the extensions for relaxing such assumptions.

Functional form assumptions  To get a closed-form solution for the equilibrium, I exploit three assumptions regarding the functional forms of production technology and costs: (1) Cobb-Douglas production function; (2) marginal cost of agents’ own effort is $c(\theta, \sum_{j \in N(i)} h_{jj}) = (\theta + \sum_{j \in N(i)} h_{jj})^{-\beta}$; (3) cost function of helping effort is $v(\sum_{j \in N(i)} h_{ij}) = \frac{1}{2} \gamma (\sum_{j \in N(i)} h_{ij})^2$. These assumptions altogether construct a tractable framework for the game-theoretic setting of team
production with network interactions.

However, the lemmas and propositions in this article do not necessarily require these functional form assumptions. To obtain the existence result in Proposition 1, the sufficient condition is that in the following system of equations:

\[ e_i = f_e(e_1, e_2, \ldots, e_N), \quad i \in \{1, 2, \ldots, N\} \]  \hfill (D1)

\[ h_{ij} = f_h(h_{12}, h_{13}, \ldots, h_{N-1N}), \quad i \in \{1, 2, \ldots, N\}, \quad j \in N(i) \]  \hfill (D2)

the \( f_e \) and \( f_h \) functions in equations (D1) and (D2) are concave in each of their argument, and the choice set of agents’ own effort and helping effort is bounded, according to Rosen (1965). Thus, we do not need the exact functional form assumptions as in the above model. If (1) the production function is concave in each agent’s own effort; (2) the marginal cost function is a concave function of the total received helping effort; and (3) the cost function of helping effort is convex, then the existence result still holds.

The assumptions needed to obtain the (weak) uniqueness result are more complicated. To get the uniqueness result, we need to use Proposition 2(2) and Proposition 2(3) to establish equations of helping efforts and reduce the number of unknown helping efforts, to ultimately get equations (A13), (A14), and (A15). Then, the uniqueness result relies on the monotonicity of the LHS and RHS of equations (A13), (A14), and (A15). In other words, we need restrictions on the functional forms so that the marginal benefit of the helping effort is strictly decreasing, and the marginal cost of the helping effort is strictly increasing, or the concavity of the payoff function. \(^*\) Then, according to the fixed-point theorem proposed in Kennan (2001), we can get the weak uniqueness of the equilibrium. Cobb-Douglas production function and other two restrictions on the cost functions provide an example to establish the uniqueness result with good tractability. The result of comparative statics is most sensitive to the setting of functional forms. While it is still possible that the team output does not strictly decrease after adding more links, given that more general functional forms are adopted, the finding regarding redundant

\(^*\) By Rolle’s theorem, the solution of such a system of equations exists. And by monotonicity, such a solution is unique.
links, namely that the team output remains exactly the same after links are added, depends crucially on the specific functional forms.

Another important assumption in the model is that agents provide helping effort and their own effort separately, so the cost of these two types of effort enters the payoff function separately. I make this assumption for the tractability of the model. If this assumption is not made, then the complexity of the relationship between own effort and helping effort makes the model highly untractable.

**Perfect substitutability of helping efforts** In the model, I assume that the helping efforts given to and received from different neighbors are perfectly substitutable. In other words, conditional on the total amount of helping effort given and received, agents are indifferent regarding the source and target of the helping effort. Therefore, the marginal cost of agents’ own effort and the cost of helping effort are both a function of the total amount of given and received helping effort. Such an assumption is used to establish Proposition 2(2), Proposition 2(3), and Proposition 3. Thus, the uniqueness result relies on this assumption. As discussed above, the existence result only depends on the curvature of the production function and cost function, and, thus, can still hold without this assumption. Since the establishment of the results of comparative statics relies on Proposition 2, it also relies on the assumption of perfect substitutability of helping efforts.

**Homogeneity assumption** I assume throughout this article that agents are the same in terms of abilities, their role in production, and the compensation they get. The results of the existence and uniqueness of equilibrium do not rely on this assumption, since they only require the curvature and monotonicity of the related functions. However, the results of comparative statics critically rely on this assumption. The redundant link under homogeneity might no longer be redundant under heterogeneity. For example, if one agent is extremely capable of helping others, then adding a link between him/her and another agent who has already been helped a lot by others may strictly increase the team’s output. In this case, such a link added is no longer redundant.