Absolute Positioning Model of Seafloor Geodetic Point with Systematic Error Difference Constraint

Yueyuan Ma\textsuperscript{a,b,*}, Yuanxi Yang\textsuperscript{b,c}, Anmin Zeng\textsuperscript{b,c}

\textsuperscript{a} Beijing Institute of Tracking and Telecommunication Technology, Beijing, China; \textsuperscript{b} Key Laboratory of Smart Earth, Beijing, China; \textsuperscript{c} Xi'an Research Institute of Surveying and Mapping

Contact: Yueyuan Ma, mayueyuanieu@163.com, Beijing Institute of Tracking and Telecommunication Technology, Beijing 100094, China
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Abstract: The measurement trajectory error, geometric error of measurement trajectory, and systematic error caused by the uncertainty of sound velocity are the main sources that affect the positioning accuracy of seafloor geodetic station. In this paper, a systematic error difference constraint model is proposed to control the influence of systematic errors. Firstly, the systematic error caused by the change in sound velocity is parameterized. Next, the error equation is jointly constructed with the parameters of systematic error and the position parameters of the seafloor geodetic station to be estimated. Then, based on the characteristics of measurement time and a slight change in the position of the surveying vessel, the constraint conditions of the systematic error difference between the two epochs are developed. Subsequently, to estimate the parameters expected to tackle the rank defect or ill-conditioned normal equations, the adjustment criterion with extra constraints is utilized. Finally, in the case of a single seafloor transponder, the simulation and actual experiment validate that the model with systematic error constraints can effectively control the systematic error influences on the seafloor positioning and overcome the ill-posed model issue, enabling more accurate positioning results.

Keywords: Seafloor Geodetic Point; Semi-parameter Model; Constraint condition; Absolute Positioning Model; Systematic Error

1 Introduction

The accurate positioning of the seafloor geodetic station is a key part of the seafloor geodetic datum determination, which is also an essential part of the national
space datum establishment (Yang et al., 2017, 2020; Yang and Qin, 2021; Liu et al., 2019). GNSS-Acoustic positioning technology is the most applicable method to determine the position of the seafloor geodetic station (Spiess et al., 1998; Matsumoto et al., 2008). By using the attitude measurement data and the offset data from the GNSS antenna to the transducer, the position of the GNSS antenna is calculated to deduce the position of the acoustic transducer. Then, the acoustic ranging data can be used to get the position of the seafloor geodetic station with the aid of adjustment processing.

The main factors impacting seafloor geodetic positioning accuracy are the ranging systematic errors resulting from the measuring trajectory and the spatiotemporal fluctuation of the sound velocity (Ma et al., 2022; Osada et al., 2003; Fujimoto et al., 1997). Utilizing GNSS post-differential and precise point positioning (PPP), especially the satellite-based PPP of the Beidou satellite navigation system (Yang et al., 2020), we can obtain the three-dimensional coordinates of the measurement vessel trajectory at sub decimeter-level (Zhang et al., 2019). The error of the position of the vessel trajectory is negligible compared to the decimeter-level error of acoustic ranging. The measuring track, mainly composed of linear (Obana et al., 2000; Yamada et al., 2002) and the circular path (Osada et al., 2003; Kussat et al., 2005; Chen and Wang, 2007), directly determines the intensity of the geometry of the observations (Xue and Yang, 2014). Symmetric trajectories can effectively weaken the influence of systemic errors, like the offset parameter (the offset between the GNSS antenna and the ship's transducer) errors and sound ray bending due to their excellent symmetry of observations (Ma et al., 2023). The geometrically well-balanced ranging
data is helpful for accurate positioning of the seafloor geodetic station because the horizontal positioning errors due to the uncertainty of sound velocity of seawater can be compensated (Fujita et al., 2006; Sato et al., 2013). The seabed geodetic station's horizontal component accuracy can be greatly improved by using simple symmetrical measurements consisting of a circular or rectangular track, but these measurements have a minor impact on the calculation of the vertical component (Yang et al., 2020). Therefore, some scholars proposed a circular track with a radius of \( \sqrt{2} \) times the depth plus an overhead cross-track, which guaranteed the accuracy of the horizontal component and restricted the vertical component (Chen et al., 2020; Watanabe et al., 2020). In addition to reducing the horizontal error caused by the sound velocity of the seawater through a symmetrical track, the horizontal positioning accuracy of the seafloor geodetic point can also be improved by correcting the horizontal non-uniformity of the sound velocity (Yokota et al., 2018; Honsho et al., 2019; Kinugasa et al., 2020). To improve the accuracy of the vertical component of the seafloor geodetic point, Zhao et al. proposed a model for determining the position of the transponder taking the waves and depth constraints into account (Zhao et al., 2016, 2018). Tomita et al. proposed a state-space model for vertical displacement detection based on an EKF solution (Tomita et al., 2019). Utilizing least-squares inversion with several observations, Sakic et al. investigated an alternative approach that notably increased accuracy (Sakic et al., 2020).

The geometry strength of the geographical locations of seafloor stations is also important in the seafloor geodetic positioning, especially in the tectonic process
research of the subduction zone and monitoring the crustal deformation (Yokota et al., 2016; Yasuda et al., 2017; Yokota and Ishikawa, 2020; Kido et al., 2011; Watanabe et al., 2014; Tomita et al., 2017). In accurate seafloor geodetic positioning or deformation monitoring, multiple acoustic transponders are usually placed symmetrically on the seafloor. Acoustic ranging is implemented to initially locate the array center (a virtual point), after which the relative changes between the array positions are estimated (Gagnon et al., 2005; Yokota et al., 2016; Watanabe et al., 2020). In order to extend the terrestrial geodetic datum network to the seafloor and ensure the positioning and navigation of underwater users, GNSS-A technology based on a single fixed station and multiple mooring stations is often applied to the actual measurement work (Yang and Qin, 2021), in which the fixed station is used for the construction of seafloor datum and the mooring station is used for the navigation of underwater users.

There are mainly two methods to control the influences of systematic errors. One is to weaken the influence of systematic error by differential observation (Xu et al., 2005). The alternative is to compensate for the effect of systematic errors by attaching influence parameters of systematic errors. The differential observation is used to decrease the systematic error, which is on the premise of the approximate observation environment of the adjacent epochs, which means that the two successive acoustic measurement errors are nearly the same. The estimated position accuracy of the seafloor geodetic station's horizontal components is enhanced by lessening the implications of the systematic errors of the adjacent epochs. To improve the accuracy of both horizontal and depth parameters simultaneously, some scholars constructed a differential
positioning algorithm with additional depth difference and horizontal distance constraints (Sun et al., 2019a). The observational model taking the systematic errors into account is usually called the model with additional parameters (Zhao and Liu, 2008) in which the systematic errors are treated as additional model parameters (Yang et al., 2020; Chen and Wang, 2007; Yang et al., 2011). The systematic errors can be calibrated in advance or processed afterward, such as the sound velocity errors, thus the observational model with measurement constraints is also helpful (Chen and Wang, 2007). It is also possible to apply a three-dimensional position estimation model that considers the coordinate component errors independent of the sound velocity profile (Yang et al. 2011). However, the initial average sound velocity assumed by this method may deviate from the measured mean sound velocity, therefore the deviation in sound velocity should be taken as the additional parameter to be estimated. If the systematic errors are treated as additional model parameters in each epoch, then a semi-parametric adjustment model could be applied to control the influence of systematic errors caused by changes in the sound velocity (Sun et al., 2019b). The key to the semi-parametric model is the determination of the positive definite matrix and the smoothing factor. Different determination methods may lead to different results (Wang et al., 2004; Tao et al., 2012).

In this paper, the systematic error caused by the change of sound velocity is parameterized in the observational model. The systematic error at each epoch is treated as an unknown parameter which will be estimated together with the position parameters. A systematic error constraint model for seafloor geodetic point absolute positioning is
constructed based on the premise that the two measurements are obtained in a very short time and the measurement errors are nearly the same. In this way, the ill-posed or the rank defect normal equation caused by too over-determined parameters can be overcome. The new model's positioning efficiency is verified through a simulation and a set of measured data in the deep sea of China.

2 Acoustic ranging observation equation

The underwater acoustic ranging observation equation, which takes into account errors in the sound velocity profile and the sound travel time, is expressed as (Xu et al., 2005):

\[ \rho_{i,j} = f(x_i, x_j) + \delta\rho_{vd,i} + \delta\rho_{vb,i} + \delta\rho_{vl,i} + \delta\rho_{vs,i} + \epsilon_{i,j}, \]  

(1)

where \( \rho_{i,j} \) denotes the round-trip underwater acoustic ranging observation between the shipborne transducer and the seafloor transponder \( j \) at the \( i \)-th epoch, \( f(x_i, x_j) \) is the linear distance between the transducer and the transponder \( j \) in the \( i \)-th measurement epoch. \( x_i \) denotes the position vector of the transducer at the \( i \)-th epoch. \( x_j \) is the position parameter vector of the transponder \( j \). \( \delta\rho_{vd,i} \) denotes the mechanical acoustic ranging errors. \( \delta\rho_{vb,i} \) means the ranging error is caused by the inaccurate measurement of the sound velocity profile. \( \delta\rho_{vl,i} \) is the ranging error of long period term caused by diurnal variation of tide and temperature etc. \( \delta\rho_{vs,i} \) denotes the short-period term ranging error caused by the internal wave. \( \epsilon_{i,j} \) denotes the random error.

2.1 Systematic error equation

In Eq. (1), the coefficients of systematic error parameters \( \delta\rho_{vd,i} \), \( \delta\rho_{vb,i} \), \( \delta\rho_{vl,i} \), and \( \delta\rho_{vs,i} \)
are 1. If there is no prior information attached, these different systematic error parameters cannot be separated. In other words, they cannot be estimated simultaneously. Therefore, these error parameters caused by the spatiotemporal fluctuation of the sound velocity can be signified by:

\[ \delta \rho_{d,j} = \delta \rho_{vd,j} + \delta \rho_{vb,j} + \delta \rho_{vl,j} + \delta \rho_{vl,j}. \]  

Thus, the observational Eq. (1) is simplified as:

\[ \rho_{i,j} = f(\mathbf{x}_i, \mathbf{x}_j) + \delta \rho_{d,j} + \epsilon_{i,j}. \]  

If the systematic error term is corrected by external calibration or empirical model in advance (Liu et al., 2006), the systematic error term \( \delta \rho_{d,j} \) can be removed from the observation model Eq. (3), and then by Taylor series expansion we obtain:

\[ \nu_{i,j} = a_{i,j} d\mathbf{x}_j - l_{i,j}. \]  

In the formula, \( a_{i,j} \) is the element of the design matrix, obtained by the first partial derivative of \( f(\mathbf{x}_i, \mathbf{x}_j) \), \( d\mathbf{x}_j \) denotes the estimated parameter vector and \( l_{i,j} = \rho_{i,j} - f(\mathbf{x}_i, \mathbf{x}_j) - \delta \rho_{d,i} \) denotes the observation vector.

The error equation composed of all observations can be abbreviated as:

\[ \mathbf{V} = A d\mathbf{X} - \mathbf{L}, \]  

where \( d\mathbf{X} \) denotes the position parameter correction vector, \( \mathbf{V} \) denotes the residual vector, \( A \) denotes the design matrix, \( \mathbf{L} \) denotes the observation vector.

According to the least square criterion, we can get:

\[ d\mathbf{X} = \left( A^T PA \right)^{-1} A^T PL, \]
where $\mathbf{P}$ is the weight matrix determined according to the measurement accuracy of the observed data. In general, the measurement accuracy of the data is featured with prior information. Due to the sailing circle mode, $\mathbf{P}$ can be regarded as a unit weight matrix in Eq. (6).

If both the sound speed perturbation $\delta \rho_{d,j}$ and the seafloor transponder location are unknown parameters, then the observational model is obtained like:

$$v_{i,j} = (a_{i,j} \ 1) \begin{pmatrix} \delta x_j \\ \delta \rho_{d,j} \end{pmatrix} - l_{i,j}, \quad (7)$$

where $l_{i,j} = \rho_{i,j} - f(x_i, x_{j,0})$.

2.2 Semi-parametric model of systematic error equation

The error equation composed of all of the acoustic ranging data can be abbreviated as (Sun et al., 2019b):

$$\mathbf{V} = \mathbf{A} \mathbf{dX} + \mathbf{S} - \mathbf{L}, \quad (8)$$

where $\mathbf{S} = (\delta \rho_{d,1} \ \delta \rho_{d,2} \ \cdots \ \delta \rho_{d,n})^T$ means the systematic error vector; $\mathbf{V}$ is the residual vector; $\mathbf{L}$ represents the observation vector.

Eq. (8) contains both position parameters of seafloor geodetic point and the systematic error parameters, namely, non-parametric components, which can be considered as a semi-parametric model (Sun and Wu, 2002). There are three position parameters correction $\mathbf{dX}$, $n$ systematic error parameters, but only $n$ observation equations in this model. Therefore, a unique solution cannot be worked out in Eq. (8).

A reasonable adjustment criterion for the semi-parametric model is (Fischer et al., 1999):
where $P$ is the corresponding weight matrix; $G$ is a given positive definite matrix; $\alpha$ is a smoothing factor, which balances data fitness and degree of the constraint.

The key to solving the semi-parametric model is to reasonably determine the positive definite matrix $G$ and smoothing factor $\alpha$. The selection of a positive definite matrix $G$ is related to specific problems (Fischer et al., 1999), and the smoothing factor $\alpha$ is also quite challenging to determine. Its criteria usually include the generalized cross-check method, L-curve method, and ABIC (Akaike, 1980; Yabuki and Matsu’ura, 1992; Ding and Jiang, 2004; Wang et al., 2004; Ding and Tao, 2014). Some in-depth discussions on the semi-parametric adjustment model of the seafloor geodetic points and the specific value of a specific positive definite matrix $G$ were given in the literature (Sun et al., 2019b).

3 The systematic error difference constraint model

As usual, we assume that the sampling interval of underwater acoustic ranging is 10 s to 20 s, the time difference between two measurements is manageable, and the position of the surveying vessel is not notably altered. Accordingly, the difference in the underwater sound velocity structure's acoustic ranging systematic errors between the two epochs is expected to be minimal and, to a certain extent, comparable.

3.1 Systematic error difference constraints

Based on the assumption, we have the following constraints:

$$\phi = V^T PV + \alpha S^T GS. \quad (9)$$
This constraint for all epochs can be written in the matrix form:

\[
\begin{pmatrix}
\delta \rho_{d,1} \\
\delta \rho_{d,2} \\
\vdots \\
\delta \rho_{d,n-1}
\end{pmatrix} 
\approx \begin{pmatrix}
\delta \rho_{d,2} \\
\delta \rho_{d,3} \\
\vdots \\
\delta \rho_{d,n}
\end{pmatrix} .
\] (10)

Likewise, assume that all systematic errors are equal within a period, that is, the systematic error of an epoch is the average of the systematic error of the period before and after, then we have:

\[
\delta \rho_{d,i} = \frac{1}{2N} \sum_{k=1}^{N} \delta \rho_{d,i-k} + \frac{1}{2N} \sum_{k=1}^{N} \delta \rho_{d,i+k} + \varepsilon ,
\] (12)

where \( N \) denotes the sliding window size.

The constraint can be written as:

\[
\begin{pmatrix}
-1 & 1 & & & & \\
-1 & 1 & & & & \\
\vdots & & & & & \\
-1 & 1 & & & & \\
\end{pmatrix}
\begin{pmatrix}
\delta \rho_{d,j-N} \\
\delta \rho_{d,j-1} \\
\vdots \\
\delta \rho_{d,j+N}
\end{pmatrix}
= 0 + \varepsilon .
\] (13)

The key to the constraint condition of the systematic errors is the choice of the sliding window size (Wang et al., 2004; Tao et al., 2012). Regardless of this, the
constraint equation of systematic error can be uniformly written in the form of a virtual observation equation, namely:

\[ V_s = RS - L_s, \quad (14) \]

where \( V_s \) denotes the residual vector; \( R \) denotes the design matrix; \( L_s \) denotes the virtual observation vector; In general, \( L_s \) is a zero vector and the corresponding weight is \( P_s \).

The rank of the design matrix of the virtual error equation formed by Eq. (11) is \( \text{rank}(R) = n - 1 < n \), and the rank of the design matrix of the error equation constructed by the same Eq. (13) is also satisfies \( \text{rank}(R) < n \). That is, the matrix \( R \) in Eq. (14) is a rank deficiency. Eq. (8) with additional constraint conditions Eq. (14), where the unique solution cannot be obtained and other constraint conditions are needed.

The long-period variation of the acoustic ranging systematic error is mainly similar to the tidal effect, and the variation has prominent diurnal and semi-diurnal characteristics. Short-period variation is mainly related to the oceanic internal waves (Spiess et al., 1998; Osada et al., 2003; Yasuda et al., 2017; Yokota et al., 2018; Matsui et al., 2019). It is noted that the systematic errors caused by sound velocity are periodic and non-periodic (Zhao S et al., 2021; Yang and Qin, 2021; Wang J et al., 2020; Sun W et al., 2019). Therefore, the systematic error is considered to be periodic, and the non-periodic part is absorbed by the parameters to be estimated in the article.

Suppose that the observation time is long enough with an integer multiple of the period. In this case, the sum of the systematic errors of all epochs can be expressed as a constant \( \frac{1}{n} \sum_{i=1}^{n} \delta \rho_{i,j} = \text{constant} \). That is, another constraint can be constructed as
follows:

\[ E^T S = W, \quad (15) \]

where, \( E^T = (1 \ 1 \ \ldots \ 1) \), \( W \) is a zeros vector.

### 3.2 The systematic error difference constraint model

Combined Eq. (8) and Eq. (14), the total error equation is:

\[
\begin{bmatrix}
V \\
V_s
\end{bmatrix} =
\begin{bmatrix}
A & I \\
0 & R
\end{bmatrix}
\begin{bmatrix}
dX \\
S
\end{bmatrix}
- \begin{bmatrix}
L \\
L_s
\end{bmatrix}. \quad (16)
\]

The corresponding weight matrix is:

\[
\bar{P} = \begin{bmatrix}
P & 0 \\
0 & P_s
\end{bmatrix}. \quad (17)
\]

where \( P_s \) is the weight matrix of \( L_s \).

The following objective function can be constructed, based on Eq. (16):

\[
\phi = V^T PV + V_s^T P_s V_s. \quad (18)
\]

By strict derivation, there are:

\[
\begin{align*}
N &= A^T PA \\
U &= A^T PL \\
M &= P + R^T P_s R - P A N^{-1} A^T P \\
S &= M^{-1}(P L + R^T P_s L_s - P A N^{-1} U) \\
dX &= N^{-1} A^T P (L - S) = N^{-1} U - N^{-1} A^T PS
\end{align*} \quad (19)
\]
The objective function with additional constraints can be constructed as:

\[ \phi = V^T PV + V_s^T P_s V_s + 2K(E^T S - W). \quad (20) \]

Through deduction, we have:

\[
\begin{align*}
M &= P + R^T P R - PAN^4 A^T P \\
K &= E^T M^{-1} \left( PL + R^T P L - PAN^4 U \right) - W \\
S &= M^{-1} \left( PL + R^T P L - PAN^4 U - E^T K \right) \\
\frac{dX}{dM} &= N^{-4} A^T P (L - S) = N^{-4} U - N^{-4} A^T PS
\end{align*}
\]

4 Experiments and analysis

The following three schemes are adopted in the test:

**Scheme 1**, a function model is used only to estimate the position parameters of the seafloor station.

**Scheme 2**, the semi-parametric model proposed by Sun et al. (2019b) is employed, which simultaneously estimates the position parameters and systematic errors of measurements.

**Scheme 3**, the difference constraint model is taken into account for simultaneous estimating the station position parameters and systematic errors, in which the weight matrix is given by experience.

To analyze the influence of systematic errors in different directions, the local coordinates (NEU) is adopted, where N denotes the northward component, E denotes the eastward component, and U denotes the zenith component.

4.1 Simulation analysis
We simulate the systematic errors like the following equation (Xu et al., 2005):

\[
\delta p_d = c_0 + c_1 \sin \left( \frac{2\pi (t-t_0)}{T_D} \right) + c_2 \sin \left( \frac{2\pi (t-t_0)}{T_w} \right) + c_3 \left( 1 - e^{-\frac{1}{2(2\pi)^2}} \right) + \varepsilon. \tag{22}
\]

In Eq. (22), there are five error types: the first one is a constant term, usually caused by the bias of the initial sound velocity. The second and the third ones are long-period and short-period terms caused by sound velocity variation. Based on the experiment conducted by Spiess in the North Pacific (Spiess, 1998) and the experiment by Osada near the Hawaiian Islands (Osada et al., 2003), the long-period term in the systematic error is similar to the tidal effect with a period of 12 h and an amplitude of about 20 cm. The cycle of short-period systematic errors is caused by internal waves with a duration from twenty minutes to several hours, which \( t_0 \) is the initial time and \( t \) is any time. The fourth one is the regional effect using a Gaussian correlation function related to the distance change which \( \mathbf{x} \) denotes the three-dimensional coordinate vector of the surveying vessel at the time \( t \), and \( \mathbf{x}' \) represents the three-dimensional coordinate vector of the seafloor transponder. The fifth one is a random error that satisfies a normal distribution.

We carried out two simulation experiments altogether: the underwater transponder's depth is 3,000 m, and the sailing circle radius is 3,000 m. The surveying vessel takes the transponder's vertical projection as the center to conduct sailing circle measurement for a round, ranging every 20 seconds. Thus, the number of samples in the first simulation experiment is 4320. Namely, the sampling time is exactly an integer period of the systematic errors. The systematic errors are simulated, based on Eq. (22), and the simulation parameters are shown in Table 1. In addition to the systematic error,
it is essential to consider the positioning error of the surveying vessel in the simulation. The transducer’s positions were noised by white noises with a standard deviation of 5 cm, 5 cm, and 10 cm in the east, north, and up directions respectively.

Table 1. System parameters in the simulation

<table>
<thead>
<tr>
<th>parameter</th>
<th>$c_0$ [cm]</th>
<th>$c_1$ [cm]</th>
<th>$c_2$ [cm]</th>
<th>$c_3$ [cm]</th>
<th>$T_w$ [min]</th>
<th>$T_D$ [h]</th>
</tr>
</thead>
<tbody>
<tr>
<td>value</td>
<td>10</td>
<td>30</td>
<td>12</td>
<td>2</td>
<td>20</td>
<td>12</td>
</tr>
</tbody>
</table>

Figure 1 shows the error of short period term caused by internal waves, the systematic error of lunisolar tides, the correlation error of regional effect, and the total error. Figure 2 shows the calculated residuals and estimated systematic errors of different schemes. Table 2 shows the positions calculated by different schemes and their inner precision, $m_0$ is posterior mean square error of unit weight, $m_v$, $m_n$, and $m_u$ is the square root of variance. Table 3 shows the residual statistic values estimated by different schemes. Table 4 shows the systematic error statistic values estimated by different schemes.

![Figure 1](image1.png)  
![Figure 2](image2.png)

Figure 1. Simulated systematic error (left: itemized systematic errors, right: total systematic errors)

Table 2. Position precision calculated by different schemes [m]

<table>
<thead>
<tr>
<th>$m_v$</th>
<th>$m_n$</th>
<th>$m_u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>value</td>
<td>value</td>
<td>value</td>
</tr>
<tr>
<td>Scheme</td>
<td>$m_0$</td>
<td>$m_e$</td>
</tr>
<tr>
<td>----------</td>
<td>--------</td>
<td>--------</td>
</tr>
<tr>
<td>Scheme 1</td>
<td>0.229</td>
<td>0.007</td>
</tr>
<tr>
<td>Scheme 2</td>
<td>0.017</td>
<td>0.001</td>
</tr>
<tr>
<td>Scheme 3</td>
<td>0.009</td>
<td>0.000</td>
</tr>
</tbody>
</table>

Table 3. Residual statistics values for different schemes estimates [m]

<table>
<thead>
<tr>
<th>Scheme</th>
<th>Min</th>
<th>Max</th>
<th>RMS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scheme 1</td>
<td>-0.420</td>
<td>0.423</td>
<td>0.228</td>
</tr>
<tr>
<td>Scheme 2</td>
<td>-0.019</td>
<td>0.020</td>
<td>0.004</td>
</tr>
<tr>
<td>Scheme 3</td>
<td>-0.008</td>
<td>0.008</td>
<td>0.001</td>
</tr>
</tbody>
</table>

Table 4. Systematic error statistics values for estimation of different schemes [m]

<table>
<thead>
<tr>
<th>Scheme</th>
<th>Min</th>
<th>Max</th>
<th>RMS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scheme 1</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>Scheme 2</td>
<td>-0.834</td>
<td>0.839</td>
<td>0.016</td>
</tr>
<tr>
<td>Scheme 3</td>
<td>-0.851</td>
<td>0.852</td>
<td>0.009</td>
</tr>
</tbody>
</table>

Figure 2. Calculation results (left: systematic errors, right: residual errors)
We can see that from the calculation:

1. The mean square error of unit weight in the three methods is 0.229m, 0.017m, and 0.009m, respectively, indicating that the systematic errors are effectively controlled in Scheme 2 and Scheme 3, and the effect of Scheme 3 is the best. In addition, it can be seen that the influence of the systematic error on the horizontal component is the same, which is consistent with the premise of the symmetrical track in the circular sailing mode. The influence of systematic errors on the vertical component is significant, thus the main effort should be in the effect controlling the systematic errors.

2. The residuals determined in Scheme 1 show a more apparent sinusoidal trend. The amplitude is about 0.420 m, which is consistent with the shape of the simulated systematic error. The difference between the two is 0.12 m, which is the same as the systematic error constant 0.12 m obtained in Eq. (22). By contrast, the residual obtained by Scheme 2 and 3 shows prominent random characteristics that the maximum residual in the Scheme 2 is 0.020 m, and the mean square error in the unit weight is 0.004 m, while the maximum residual in the Scheme 3 is 0.008 m, and the mean square error in the unit weight is 0.001 m. It shows that both the accuracy of the Schemes 2 and 3 is significantly improved, and the latter is improved more than the former.

3. From the perspective of estimated systematic errors, we find that Scheme 1 can only estimate the position parameters but cannot count the systematic errors. By contrast, both the semi-parametric model and the systematic error constraint model can estimate the systematic errors. The trend of the systematic errors is very similar to the simulated ones, but the amplitudes are increased, reaching about 0.83 m and 0.85 m, respectively.

The number of samples in the second simulation experiment is 360. Namely,
the sampling duration is not an integer period of the systematic error, and other conditions remain the same. Table 5 shows the positions calculated by different schemes and their standard deviations. Table 6 and Table 7 show the residual statistics and the systematic error statistics estimated by different schemes respectively. Figure 3 shows the calculated residuals and estimated systematic errors of different schemes.

Table 5. Position precision calculated by different schemes [m]

<table>
<thead>
<tr>
<th>Scheme</th>
<th>$m_0$</th>
<th>$m_c$</th>
<th>$m_p$</th>
<th>$m_u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scheme 1</td>
<td>0.089</td>
<td>0.009</td>
<td>0.009</td>
<td>0.007</td>
</tr>
<tr>
<td>Scheme 2</td>
<td>0.009</td>
<td>0.001</td>
<td>0.001</td>
<td>0.001</td>
</tr>
<tr>
<td>Scheme 3</td>
<td>0.001</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
</tbody>
</table>

Table 6. Residual statistics values for different schemes estimates [m]

<table>
<thead>
<tr>
<th>Scheme</th>
<th>Min</th>
<th>Max</th>
<th>RMS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scheme 1</td>
<td>-0.161</td>
<td>0.160</td>
<td>0.088</td>
</tr>
<tr>
<td>Scheme 2</td>
<td>-0.020</td>
<td>0.021</td>
<td>0.004</td>
</tr>
<tr>
<td>Scheme 3</td>
<td>-0.008</td>
<td>0.008</td>
<td>0.001</td>
</tr>
</tbody>
</table>

Table 7. Systematic error statistics values for estimation of different schemes [m]

<table>
<thead>
<tr>
<th>Scheme</th>
<th>Min</th>
<th>Max</th>
<th>RMS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scheme 1</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>Scheme 2</td>
<td>-0.390</td>
<td>0.388</td>
<td>0.187</td>
</tr>
<tr>
<td>Scheme 3</td>
<td>-0.422</td>
<td>0.420</td>
<td>0.204</td>
</tr>
</tbody>
</table>
When the sampling number is the non-integral period of the systematic error, we can find the following facts from the above charts.

1. The mean square error of unit weight in the three methods is 0.089m, 0.009m, and 0.001m, respectively. Scheme 2 and scheme 3 can still effectively control the effects of the system errors, and the effect of scheme 3 is smaller.

2. From the perspective of residuals, the maximum residual in scheme 2 is 0.021m, and the RMS is 0.004m; In scheme 3, the maximum residual is 0.008m, and the mean square error of unit weight is 0.001m, which is almost consistent with the case that the number of samples is the integer period.

3. From the perspective of systematic errors, the RMS in Scheme 2 and Scheme 3 is 0.187m and 0.204m, respectively, which again indicates that Scheme 3 can control...
the effects of systematic errors more effectively than Scheme 2.

4.2 Actual observation analysis

The experiment was carried out in the South China Sea (19°50'11.66"N, 118°27'17.34"E) during July 13-15, 2019. The "XIANG YANG HONG 18" scientific research vessel participated in the data collection. The average depth of the test area is 3000 m, and the seafloor is flat, a transponder was set on the seafloor. Due to the control of surveying vessels and other reasons, the actual measurement is not a standard circle. Since no external reference value with higher accuracy can be found, the figure is divided into three sections, indicated as A, B and C, and the calculation results of the observed figures in each section are taken as reference, as is shown in Figure 4. The shipping speed was about 4 nautical miles/hour, and the measurement time was about 1.5 h. The acoustic ranging was conducted per 8 s, thus obtaining 684 acoustic ranging measurements. GNSS antenna's position was collected by a GNSS satellite-based differential receiver with a sampling rate of 1 Hz. The surveying vessel's attitude was obtained with the aid of the attitude measuring equipment. The sampling rate was 5 Hz. Before measurement, the shipborne transducer coordinates, GNSS antenna, and attitude equipment in the vessel body coordinate system were strictly measured. In this way, it is very convenient to calculate the position of the acoustic transducer. The CTD was used to measure the sound velocity profile, as shown in Figure 5. The ray-tracing algorithm based on constant-gradient can work out the distance between the transducer and seafloor geodetic point through the measured sound velocity profile data and travel-time observation data.
Table 8. Coordinates and inner precision calculated by different schemes [m]

<table>
<thead>
<tr>
<th>Path</th>
<th>Scheme</th>
<th>Coordinates</th>
<th>Precision</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>E</td>
<td>N</td>
</tr>
<tr>
<td>A</td>
<td>Scheme 1</td>
<td>0.333</td>
<td>-0.168</td>
</tr>
<tr>
<td></td>
<td>Scheme 2</td>
<td>0.307</td>
<td>-0.232</td>
</tr>
<tr>
<td></td>
<td>Scheme 3</td>
<td>0.320</td>
<td>-0.177</td>
</tr>
<tr>
<td>B</td>
<td>Scheme 1</td>
<td>0.109</td>
<td>-0.367</td>
</tr>
<tr>
<td></td>
<td>Scheme 2</td>
<td>0.174</td>
<td>-0.193</td>
</tr>
<tr>
<td></td>
<td>Scheme 3</td>
<td>0.074</td>
<td>-0.411</td>
</tr>
<tr>
<td>C</td>
<td>Scheme 1</td>
<td>-0.060</td>
<td>-0.518</td>
</tr>
<tr>
<td></td>
<td>Scheme 2</td>
<td>-0.103</td>
<td>-0.115</td>
</tr>
<tr>
<td></td>
<td>Scheme 3</td>
<td>-0.092</td>
<td>-0.376</td>
</tr>
</tbody>
</table>

Table 9. Residual statistics for different schemes estimates [m]

<table>
<thead>
<tr>
<th>Path</th>
<th>Scheme</th>
<th>Min</th>
<th>Max</th>
<th>RMS</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Scheme 1</td>
<td>-0.802</td>
<td>0.841</td>
<td>0.272</td>
</tr>
<tr>
<td></td>
<td>Scheme 2</td>
<td>-0.494</td>
<td>0.457</td>
<td>0.156</td>
</tr>
<tr>
<td></td>
<td>Scheme 3</td>
<td>-0.229</td>
<td>0.239</td>
<td>0.078</td>
</tr>
<tr>
<td></td>
<td>Scheme 1</td>
<td>Min</td>
<td>Max</td>
<td>RMS</td>
</tr>
<tr>
<td>---</td>
<td>----------</td>
<td>------</td>
<td>------</td>
<td>------</td>
</tr>
<tr>
<td>A</td>
<td>Scheme 1</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td></td>
<td>Scheme 2</td>
<td>-0.765</td>
<td>1.015</td>
<td>0.506</td>
</tr>
<tr>
<td></td>
<td>Scheme 3</td>
<td>-1.196</td>
<td>1.149</td>
<td>0.388</td>
</tr>
<tr>
<td>B</td>
<td>Scheme 1</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td></td>
<td>Scheme 2</td>
<td>-0.728</td>
<td>0.763</td>
<td>0.370</td>
</tr>
<tr>
<td></td>
<td>Scheme 3</td>
<td>-0.899</td>
<td>0.702</td>
<td>0.232</td>
</tr>
<tr>
<td>C</td>
<td>Scheme 1</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td></td>
<td>Scheme 2</td>
<td>-0.491</td>
<td>0.464</td>
<td>0.192</td>
</tr>
<tr>
<td></td>
<td>Scheme 3</td>
<td>-0.663</td>
<td>0.516</td>
<td>0.189</td>
</tr>
</tbody>
</table>

Table 10. Systematic error statistics estimated by different methods [m]

![Histograms](image.png)

(a) Residual by path A
Figure 6. Residuals calculated by different schemes

(a) Systematic error by path A

(b) Systematic error by path B
Table 8 shows the coordinate and inner precision calculated by the different schemes. Table 9 and Table 10 show the residual statistics and systematic error statistics estimated by different methods. Figure 6 and Figure 7 show the residual values and systematic error calculated by different schemes, respectively. From the charts above, the conclusions can be drawn as follows:

(1) Under different trajectories, in Scheme 1, the maximum differences of estimation results are 0.393 m, 0.350 m and 0.165 m, and the maximum difference of inner precision is 9.1 cm, 31.1 cm and 18.9 cm for N, E and U component respectively. In Scheme 2, the maximum differences in estimation results are 0.410 m, 0.117 m and 0.651 m, and the maximum difference in inner precision is 7.7 cm, 32.5 cm and 10.0 cm for N, E and U component respectively. In Scheme 3, the maximum differences of estimation results are 0.412 m, 0.234 m and 0.163 m, and the maximum difference in inner precision is 9.0 cm, 30.9 cm and 18.9 cm for N, E and U component respectively. The results indicate that it is likely to obtain more accurate coordinate estimates and better accuracy of point position by optimizing the positioning model and modifying the algorithm.
(2) Under different trajectories, the residual RMS calculated by Scheme 1 is 0.272 m, 0.165 m, 0.132 m; the residual RMS calculated by Scheme 2 is 0.156 m, 0.109 m, 0.073 m, and the residual RMS calculated by Scheme 3 is 0.078 m, 0.047 m, 0.038 m. In addition, it can be seen from Figure 6 that under different trajectories, the residual errors calculated by Scheme 2 and Scheme 3 are consistent with normal distribution, but the RMS value of Scheme 3 is smaller, indicating that Scheme 3 can constrain the systematic error better.

(3) Under different trajectories, the variation trend of the systematic error estimated by Scheme 2 and Scheme 3 is almost accorded with that of the residual error of Scheme 1, showing an obvious change of sine and cosine, which indicates that there is a periodic change. As can be seen from the figure, the systematic error figure estimated by Scheme 3 is more approximate to the trend of Scheme 1, indicating that the systematic error estimated by Scheme 3 is closer to the actual systematic error, which again indicates that Scheme 3 can better constrain the systematic error, so as to diminish the impact of the systematic error.

5 Conclusions

The systematic error difference constraint model has been presented in this paper to control the influence of systematic errors on seafloor geodetic point positioning. The simulation and actual example have verified the effectiveness of the proposed method. The conclusions can be drawn as follows:

1. The periodic variation of the systematic error exists in the underwater acoustic measurements. The accuracy of seafloor stations' horizontal component isn't
substantially affected by the systematic error, but their vertical component is.

2. The systematic error difference constraint model can effectively restrain the systematic error influence on the location parameter estimates of the seafloor stations, and the estimation solution is stable compared to the semi-parametric model.

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Author contributions
All authors contributed to the study's conception and design. Yueyuan Ma, Anmin Zeng undertook the formula derivation, model construction. The first draft of the manuscript was written by Yueyuan Ma, and Yuanxi Yang commented and modified on previous versions of the manuscript. All authors read and approved the final manuscript.

Data availability
The data sets analyzed during the current study are not publicly available due to the management regulations of relevant organizations, but all data included in this study are available upon reasonable request from the corresponding author.

References


