Feasible Joint Distributions over Actions & Stopping Times

Andrew Koh†
MIT

Sivakorn Sanguanmoo‡
MIT

October 9, 2023

Abstract

A decision maker (DM) faces a binary choice problem and pays a constant cost per unit time to acquire more information. We characterize the joint distributions over the DM’s actions and stopping times which can arise through information in terms of a finite set of linear inequalities. We then study optimal dynamic information structures for a designer with preferences over the DM’s action and stopping time: when preferences over them are additively separable, optimal solutions take a “bang-bang” structure; when preferences over them are supermodular, optimal solutions take a “pure bad news” structure; when preferences are supermodular and concave, optimal solutions take an “almost geometric” structure in which continuation beliefs exhibit an initial jump before bad news arrives at a constant rate which leads continuation beliefs to drift gradually upwards.

1 Introduction

A decision maker (DM) faces a binary choice problem and, at each time step, decides between waiting to acquire information or acting immediately. This problem of optimal stopping under sequential sampling was studied influential paper of Wald (1947). The subsequent literature has typically held fixed the dynamic information structure (i.e., the signal process) and focused on understanding the DM’s optimal stopping policies (Arrow, Blackwell, and Girshick, 1949; Chernoff, 1961). A natural question, however, is how the DM’s optimal action and stopping time changes with the dynamic information structure.

Our paper gives a partial answer to this question by characterizing the joint distributions over actions and stopping times which can arise from any dynamic information structure in terms
of a finite number of linear inequalities. The key for this characterization is the observation that, while the space of dynamic information structures is large, it is sufficient to focus on a special class of information structures which, at any point in time, either induces the DM to stop at extremal beliefs—those which make the DM either certain of the state, or indifferent across actions—or induces the DM to continue waiting which, in turn, corresponds to a unique continuation belief. For any arbitrary dynamic information structure, we can always modify it so that (i) the distribution of DM’s optimal action and stopping time is preserved; and (ii) the DM’s continuation and stopping incentives are also preserved.

We then use this characterization to solve for designer optimal dynamic information structures when a designer has preferences over both the DM’s action (as in Kamenica and Gentzkow (2011)) and stopping time (as in our companion paper Koh and Sanguanmoo (2022)). When the designer’s preference over actions and stopping times are additively separable (but potentially nonlinear), we show that optimal information structures take a “bang-bang” form: it either gives the DM full information at random times, or persuades the DM at a fixed, deterministic time. When the designer has supermodular preferences over actions and stopping times, we show that the optimal information structure either provides full information at random times, or takes on a “pure bad news” structure: there exists a fixed, deterministic time $T$ so that the only information the DM receives before time $T$ is a bad-news signal which makes DM certain the state is $0$. Such information structures facilitate the designer’s objective in two ways: first, it increases the DM’s continuation value which allows the designer to make the time-$T$ information structure less informative—hence making the DM take action 1 more often; second, it generates increasing belief paths so that conditional on not receiving bad news, the DM is more confident the state is $1$. This makes persuading the DM at time $T$ into action 1 more likely. Under stronger assumptions on the designer’s preferences, we strengthen this to show that such informational structures are “almost-geometric”: the bad news signal arrives at a constant rate in the interior (i.e., for times $1 < t < T$). In particular, when the prior is misaligned with the designer (i.e., the DM takes action 0—which is not preferred by the designer—under the prior), at time $t = 1$, bad news arrives at a higher rate so that conditional on not receiving bad news, the DM’s continuation belief jumps to make her approximately indifferent between actions; this is approximately the static one-period Bayesian persuasion information structure. For subsequent periods, bad news arrives at a constant rate. This, in turn, generates value for information since for those periods, the DM’s optimal action under her belief is to take action 1. Continuation beliefs thus exhibit an initial jump followed by a gradual increase until the terminal time.

Beyond the setting of information design, our results contribute to an informationally robust view of optimal sequential sampling.¹ For instance, an analyst might observe choice and stopping time data but not the dynamic information structure the DM faces. Our results pave the way to understanding the kinds of choice and stopping time behavior which can be be rationalized by Bayesian updating and optimal stopping.²

¹See, e.g., Bergemann and Morris (2013).
²This is in the spirit of Fudenberg, Strack, and Strzalecki (2018) who study the uncertain-difference drift diffusion model, and Fudenberg, Newey, Strack, and Strzalecki (2020) who show how to test drift-diffusion models.
2 Model

2.1 Environment. There is a binary state and binary action space $\Theta = A = \{0, 1\}$. We let $\mu_t \in [0, 1]$ denote the DM’s time–$t$ beliefs that the state is 1. For simplicity, we will assume that the DM pays a constant per-period cost $c > 0$ which is also adopted in recent papers on optimal stopping and dynamic information acquisition/design (Fudenberg, Strack, and Strzalecki, 2018; Che and Mierendorff, 2019). We later discuss when this can be generalized to additive separable costs via an appropriate transformation.

The DM’s utility from stopping at time $\tau$ and taking action $a$ under state $\theta$ is thus given by

$$u(a, \theta) - c\tau.$$

For an arbitrary decision problem $u$, we will assume without loss of generality that the DM’s optimal action is 1 whenever $\mu \geq \bar{\mu}$, and 0 whenever $\mu < \bar{\mu}$.

2.2 Dynamic information structures. A dynamic information structure is a series of functions

$$\{p_t(\cdot | \theta, H_t)\}_{t \in \mathcal{T}, \theta \in \Theta, H_t \in \mathcal{H}},$$

where $p_t(m | \theta, H_t)$ denotes the probability that message $m \in M$ is realised given that the state is $\theta$ and given the history of past message realizations $H_t \in \mathcal{H}_t := \prod_{s=1}^t M$. We shall assume that our message space $M$ is large but finite and the same across every period. Let $I$ denote the space of all dynamic information structures where we use $I$ to denote a typical member.

2.3 Measures under different dynamic information structures. We will often vary the dynamic information structure to understand how variables of interest (e.g., probabilities of histories, incentive compatibility constraints etc.) change. To this end, we will use $\mathbb{E}^I(\cdot)$, $\mathbb{E}^I(\cdot | \cdot)$, $\mathbb{P}^I(\cdot)$, and $\mathbb{P}^I(\cdot | \cdot)$ to denote the unconditional and conditional expectations and probabilities under dynamic information structure $I \in I$. For instance, $\mathbb{P}^I(H_t)$ and $\mathbb{P}^{I'}(H_t)$ are the unconditional probabilities that history $H_t \in \mathcal{H}_t$ realizes under dynamic information structures $I$ and $I'$ respectively. Throughout this paper we use superscripts to track dynamic information structures, and subscripts to track time periods.

2.4 DM’s optimization problem. Facing dynamic information structure $I \in I$, the DM solves the following optimization problem:

$$\sup_{\tau, a_\tau} \mathbb{E}^I[u(a_\tau, \theta) - c\tau],$$

where $\tau$ is a stopping time and $a_\tau$ is a (stochastic) action under the natural filtration. Throughout we will assume that the DM breaks indifferences in favour of not stopping to ensure that

without constant boundaries.
the set of feasible stopping times is closed.

2.5 Feasible distributions of actions and stopping times. For a given information structure \( I \in \mathcal{I} \) (and tie-breaking rule on actions), this induces the DM to stop at random times and take random actions. Let the joint distribution of the DM’s action and stopping time be \( d(I) \in \Delta(A \times \mathcal{T}) \). For a given \( d \), we let \( \tau \) denote the random variable distributed according to \( \text{marg}_a d \). We say that \( d \) is feasible if there exists an information structure \( I \in \mathcal{I} \) and tie-breaking rule for actions such that \( d(I) = d \).

2.6 Information Design. We will often be interested in the maximization problem of a designer who cares about both when the DM stops, and what action the DM takes when she does. The designer’s payoff from the DM stopping at time \( \tau \) and taking action \( a \) is given by \( f(a, \tau) \) so it solves

\[
\sup_{I \in \mathcal{I}} \mathbb{E}[f(a, \tau)].
\]

We will assume that the sender’s preference over DM’s action is monotone so that \( f(1, t) \geq f(0, t) \) for all \( t \in \mathcal{T} \).

3 Reduction and Optimal Information Structures

Our first result shows that any feasible distribution over actions and stopping times can be implemented by an information structure under which stopping beliefs take values in the set \( \{0, \bar{\mu}, 1\} \). This generalizes the “reduction principle” which studies only implementable stopping times (Koh and Sanguanmoo, 2022, Proposition 1).

**Definition 1.** An information structure \( I \) is extremal and obedient if it is supported on a set of distinct messages \( \{m_\mu\}_{\mu \in \{0, \bar{\mu}, 1\}} \cup \{m_0\} \) such that

(i) (Extremal beliefs) \( \mathcal{F}^{\mathcal{I}}(\theta = 1|H_t = ((m_\theta)^{t-1}, m_t = m_\mu)) = \mu \) for all \( \mu \in \{0, \bar{\mu}, 1\} \)

(ii) (Obedience) DM (weakly) prefers to continue paying attention on every history \( H \in \{((m_\theta)^t : t \in \mathcal{T}\} \) which is reached with positive probability.

Part (i) states that stopping beliefs lie in the set \( \{0, \bar{\mu}, 1\} \). This is a weaker condition than the “full revelation” information structures employed in Koh and Sanguanmoo (2022) which required the DM to be certain about the state when she stops (i.e., beliefs lie in the set \( \{0, 1\} \)).

Part (ii) states that the DM weakly prefers to continue paying attention for another period upon receipt of the message \( m_\theta \). Let \( \mathcal{I}^{\text{EXT}} \) denote the space of extremal and obedient structures.

---

3This is not necessary; we make this assumption for simplicity so that we always tiebreak in favor of the designer.

4This reduction to extremal beliefs which leave the DM either certain of the state, or indifferent between actions is related to the “extremal segmentations” employed by Bergemann, Brooks, and Morris (2015) in the static monopoly setting with unit demand. In our dynamic setting this reduction preserves the joint distribution of actions and state and hence continuation incentives.
**Proposition 1.** Any feasible joint distribution \( d \in \Delta(A \times T) \) is implementable through an extremal and obedient information structure \( I \in I^{\text{EXT}} \) with an appropriate tie-breaking rule.

We defer the proof of Proposition 1 to Appendix B where we state and prove a more general version for an arbitrary number of states and actions. We sketch the main intuition here. Fix any dynamic information structure which induces a joint distribution \( d \in \Delta(A \times T) \). From a similar argument as in the reduction principle from Koh and Sanguanmoo (2022), we can always collapse continuation nodes into a single node—by the convexity of the max operator and Jensen’s inequality, this preserves continuation incentives. Now for any fixed time \( t \), consider the distribution of stopping beliefs at \( t \) which we denote with \( \mu_t^S := \mu_t|\{\tau = t\} \in \Delta(\Theta) \). Observe that for any given tie-breaking rule, this induces a distribution of actions over \( A \). We then conclude by arguing that we can replicate this distribution of actions over \( A \) with an alternate stopping belief \( \mu_t^C \) supported on the set \( \{0, \bar{\mu}, 1\} \) while, at the same time, preserving the (i) martingale constraint on beliefs; (ii) the continuation belief \( \mu_t^C = \mu_t^C \); and (iii) the distribution of the stopping time:

\[
\mathbb{E}[\mu_t|\tau > t - 1] = \mathbb{P}(\tau = t)\mathbb{E}[\mu_t^S] + \mathbb{P}(\tau > t)\mathbb{E}[\mu_t^C]
= \mathbb{P}(\tau = t)\mathbb{E}[\mu_t^S] + \mathbb{P}(\tau > t)\mathbb{E}[\mu_t^C]
= \mathbb{E}[\mu_t^C|\tau > t - 1].
\]

Note that \( \mathbb{E}[\mu_t^S] = \mathbb{E}[\mu_t^C] \) hence the modified stopping belief is in effect mean-preserving. Furthermore, note that the obedience constraint for all previous periods is preserving since the continuation value of information weakly increases. We can then proceed with this modification for all times so that under the modified information structure, all stopping beliefs are supported on \( \{0, \bar{\mu}, 1\} \).

A useful implication of Proposition 1 is that the set \( \mathcal{D} \) is characterized by a finite number of linear inequalities: since stopping beliefs map uniquely (for a given tiebreaking rule) to actions, \( d \) is feasible if and only if we can find a joint distribution over stopping beliefs and stopping times which fulfils (i) martingale; (ii) total probability; and (iii) obedience constraints while being consistent with \( d \). We will make use of this extremal and obedient reduction to understand feasible joint distributions over actions and stopping times. Our first result characterizes the designer-optimal dynamic information structure whenever preferences over actions and stopping times are additively separable.

**Proposition 2** (Bang-bang solution when attention and persuasion are additively separable). Suppose that \( f(a, \tau) = a + h(\tau) \) for some increasing function \( h : T \to \mathbb{R}_{\geq 0} \). Then an optimal dynamic information structure either:

(i) (Pure attention capture) is equivalent to the designer-optimal when \( f(a, \tau) = h(\tau) \); or

(ii) (One-shot persuasion) there exists some time \( T \) such that the sender only reveals information at time \( T \).
Proposition 2 states that when the DM’s action and attention enters additively into the designer’s payoff, optimal dynamic information structures are particularly stark. In the pure attention case (part (i)) they focus on extracting attention in the sense that the optimal dynamic information structure coincides with that if the designer’s value function were simply $h(\tau)$—this problem was studied in our companion paper Koh and Sanguanmoo (2022). Note that for this case, the designer always provides full information at stopping nodes and never finds it optimal to “persuade” the DM into taking action 1 when it is in fact state 0. Hence, the probability that the DM takes action 1 is simply the prior and the designer’s value function is

$$\mathbb{E}^I[f(a, \tau)] = \mathbb{E}^I[h(\tau)] + \mu_0.$$ 

Alternatively, the designer only provides information at a fixed and deterministic time $T$ (part (ii)) so that her value function is

$$\mathbb{E}^I[f(a, \tau)] = h(T) + \mathbb{E}^I[a_\tau]$$

and it aims to persuade the DM into taking action 1 when it is fact state 0 while leaving the DM enough value such that she finds it worthwhile to pay attention up to time $T$. In the case where $h$ is linear, the solution is even starker as the next example illustrates:

**Example 1** (Bang-bang solution under linear and additive revenue streams). Consider a retail platform which receives two revenue streams: the first from advertisements which we assume is linearly increasing in the time users spend on the platform, and the second from influencing the user into buying an expensive product because it obtains commissions when the user does so. The platform’s value function can thus be written as $f(a, \tau) = \tau + \lambda a$ where $\lambda \geq 0$ measures the degree to which the platform can monetize the users’ action (via commissions) relative to monetizing attention (via advertisements). We assume, for simplicity, that $\phi(\mu_0)/c$ is an integer. The platform’s solution is then especially stark: it either gives the user full information at time $T = \phi(\mu_0)/c$, or persuades the user in period 1. More precisely, Define

$$\lambda^*(\mu) = \begin{cases} 
1/c & \text{if } \mu_0 > 1/2 \\
1/(1 - 2\mu_0)c & \text{if } \mu_0 \leq 1/2.
\end{cases}$$

The platform’s optimal information structure is as follows:

(i) If $\lambda < \lambda^*$ the platform will send full information only once at time $\frac{\phi(\mu_0)}{c}$.

(ii) If $\lambda \geq \lambda^*$, then the sender will one-period persuade at time 1: the information structure at time 1 leaves the user value of information exactly equal to $c$; in limit as we shrink time intervals and rescale costs accordingly, this coincides with the static Bayesian persuasion solution.

This dynamic information structure is illustrated in Figure 1 where the left hand side shows part (i) and the right hand side shows part (ii).
Figure 1: Illustration of bang-bang solution: full information vs static persuasion

We have given a bang-bang characterization of the case in which the designer’s preferences are additively separable. In many settings, however, it is natural to suppose that a designer has more complicated joint preferences over actions and stopping times. We outline out several such examples.

(i) **Advisor** An advisor/consultant is paid for her time, but receives a $B$ percent bonus if the DM takes a certain action (e.g., chooses to make the investment). The consultant’s payoff function is then $f(a, \tau) = h(\tau)[1 + aB]$ where the strictly increasing function $h(\tau)$ is the payment schedule for her time.

(ii) **Judge-Prosecutor** A prosecutor would like to win her trial, as in Kamenica and Gentzkow (2011). However, suppose that conditional on winning, she would like to draw out the trial—perhaps because this enhances its publicity and hence her career prospects. Her payoff is then $f(a, \tau) = h(\tau)a$ for some strictly increasing function $h$.

In both examples, the designer’s preferences are supermodular in the usual order over $A$ and $T$: $f(1, t) + f(0, s) \geq f(1, s) + f(0, t)$ for all $t \geq s$. Our next result pins down the properties of designer-optimal structures under supermodular preferences.

**Proposition 3** (Supermodular preferences). Suppose $f$ is supermodular. An optimal extremal and obedient information structure either

(i) (Stopping signals are fully revealing) $p_{\mu, t} = 0$ for all $t \in T$; or

(ii) (Pure bad news) There exists a unique terminal time $T$ so that $P^f(\tau > T) = 0$. Furthermore, for times $t < T$, $p_{\mu, t} = p_{\mu, t} = 0$ and for times $\{t < T : p_{0, t} > 0\}$ the receiver is indifferent between continuing and stopping.

Proposition 3 states that when $f$ is supermodular, we either have that stopping signals are fully revealing, or the information structure takes a pure bad news form: before the terminal time $T$, the DM only stops when she receives a bad news signal which makes her certain that the state is 0. Furthermore, for all times $t$ at which the DM has positive probability of receiving the bad news signal, if she does not receive it, she is exactly indifferent between stopping and
continuing. At the terminal time $T$, the DM stops with probability 1 with terminal beliefs in the set $\{0, \bar{\mu}, 1\}$. This is illustrated in Figure 2 where we use $p_{m,t}$ to denote the unconditional probability that the DM stops at time $t$ with belief $m$, and $\mu^C_t := \mu_t | \tau > t$ as the continuation belief at time $t$ conditional on not yet receiving a stopping message.

The broad intuition for the pure bad news structure is as follows: from supermodularity, the sender would like DM to take action 1 at late times, and action 0 at earlier times. The pure bad news signal structure facilitates this in two ways. First, it pushes up the value of attention in earlier periods: for times $B < C < 1$, suppose that there is some chance of bad news arriving at time $C (p_0 > 0)$. If $\mu^C > 0$, this pushes up the continuation value at all times $s < t$ which, in turn, allows the information structure in the terminal stage to be skewed more towards the beliefs $\bar{\mu}$. This persuades the DM into taking action 1 more frequently. Second, the pure bad news structure makes continuation beliefs increase over time: conditional on not learning that the state is 0, the DM becomes more optimistic that the state is 1 so that $\{\mu^C_t\}_{t=1}^{T-1}$ is increasing. This increases the probability that the DM can be persuaded into action 1 at time $T$. The formal proof of Proposition 3 is fairly involved, and is deferred to Appendix A.

Definition 2. An information structure is almost-geometric bad news if it there exists a terminal time $T \in \mathcal{T}$ such that

(i) $p_{1,t} = 0$ for all $t < T - 1$, and $p_{0,t} = 0$ for all $t < T$

(ii) For each $2 \leq t < T$, the conditional probability at time $t - 1$ that bad news is revealed at time $t$ is $c / [u(0,0) - u(0,1)]$. Furthermore, DM is indifferent between continuing or stopping at every time $t < T$.

We call it almost-geometric because for all times except the boundaries (i.e., $2 \leq t \leq T - 1$), the conditional probability of bad news arriving is constant.

Proposition 4. Suppose that $f(\cdot, \cdot)$ is supermodular, $f(\cdot, 1)$ is concave, and $f(\cdot, 0)$ is linear. Then an almost-geometric bad news information structure is optimal.
Proposition 4 sharpens Proposition 3 when we impose more assumptions on the functional form of \( f \). In particular, even in the case of fully revealing stopping signals (part (i) of Proposition 3), Proposition 4 states that for all times \( t < T - 1 \), the DM receives only pure bad news signals; only at times \( t = T - 1 \) and \( t = T \) can she receive a pure good news signal which induces her into taking action 1. Furthermore, Proposition 4 states that with the exception of the boundaries, bad news arrives at a constant rate which, conditional on not arriving, leaves the DM exactly indifferent between stopping and continuing to pay attention.

Proposition 4 also implies that whenever the DM’s prior is misaligned with the designer i.e., \( \mu_0 < \bar{\mu} \) so that her action under the prior is not action 1, then conditional on not receiving bad news at time 1, the DM’s beliefs jump from the prior \( \mu_0 \) to \( (\mu_0 / \bar{\mu} - c) \) at time 1. From there, bad news arrives at a constant conditional rate \( c/[u(0, 0) - u(0, 1)] \) and beliefs gradually drift upwards until the terminal time \( T \). The next example illustrates this.

**Example 2.** Suppose that the sender’s payoff is \( f(a, t) = a(t + \gamma) \) for some constant \( \gamma > 0 \). Note that \( f(1, \cdot) \) is linear in \( t \), and \( f(0, t) = 0 \) so the conditions in Proposition 4 apply. We suppose that the receiver’s payoff from taking action \( a \) at state \( \theta \) at time \( t \) is quadratic loss \(- (a - \theta)^2 - c \cdot t \) so she obtains a payoff of 0 when she matches her action to the state, and \(-1 \) otherwise. We will suppose that (i) \( 1 / c \) is an integer; and (ii) if \( \mu_0 \geq 1 / 2 \) then the exists some positive integer \( T^{\text{MAX}} \) such that \( \mu_0 = (1 - c)^{T^{\text{MAX}}} \), if \( \mu_0 < 1 / 2 \) then there exists some positive \( T^{\text{MAX}} \) such that \( \frac{\mu_0}{2\mu_0 - c} = (1 - c)^{T^{\text{MAX}}} \). These conditions simplify the form of our solution and arises solely from the discreteness of time steps; if they do not hold, the qualitative properties of this example would continue to obtain.\(^5\)

**Type (i): Full information.** In the case where the receiver always obtains full information eventually, the optimal dynamic information structure has terminal time at \( T^{\text{MAX}} \) and take the following form: for all times \( t < T^{\text{MAX}} \), the receiver has conditional probability \( c \) of receiving perfect bad news. Doing so leaves her exactly indifferent between stopping and continuing to pay attention. At time \( t = T^{\text{MAX}} \), the receiver then obtains a fully-revealing signal. The ex-ante probability that the receiver takes action 1 is therefore \( \mu_0 \), but conditional on taking it, she only does so at time \( T^{\text{MAX}} \). The designer’s utility is thus \( \mu_0(T^{\text{MAX}} + \gamma) \).

**Type (ii): Partial persuasion.** In the second case, the unconditional probability that the DM takes action 0 is strictly greater than \( \mu_0 \)—persuasion is partial in this sense. In particular, it continues to take a perfect bad-news form in which at each time-step, DM is exactly indifferent between stopping and continuing. The belief path of the DM is thus exactly equal to that of the full information case, except that the information structure is cut short: there is some terminal time \( T \leq T^{\text{MAX}} \) at which the designer partially persuades the DM so that the ex-ante probability that the DM takes action 1 is

\[
\mathbb{P}^T(a_c = 1) = \begin{cases} 
(1 - c)^T & \text{if } \mu_0 \geq 1/2, \\
(2\mu_0 - c)(1 - c)^{T-1} & \text{if } \mu_0 < 1/2,
\end{cases}
\]

\(^5\)In particular, in the continuous time limit (as we simultaneously \( c \) small and shrink time intervals), none of these conditions would be present.
and we note that for both \( \mu_0 \geq 1/2 \) and \( \mu_0 < 1/2 \) this is strictly decreasing in \( T \). This is because for a smaller \( T \), the DM pays less waiting cost, and hence requires less value of information. This gives the designer leeway to garble information so that the DM takes action 1 more often. Further note that (i) this obtains the minimum \( \mathcal{T} = \mathcal{T}^{MAX} \) at which point \( P_{T^{MAX}}(a_r = 1) = \mu_0 \) so this coincides with the full information case; and (ii) this obtains the maximum when \( T = 1 \) at which point \( P_{T^{MAX}}(a_r = 1) = (2\mu_0 - c) \). For small \( c \), this coincides with the Bayesian persuasion solution of (Kamenica and Gentzkow, 2011).

From Proposition 4, we know that the optimal information structure must be within this class. Our problem then boils down to choosing when to terminate the information structure i.e., it solves

\[
\begin{cases}
\max_{T \leq \mathcal{T}^{MAX}} (T + \gamma)(2\mu_0 - c)(1 - c)^{T-1} & \text{if } \mu_0 \geq 1/2, \\
\max_{T \leq \mathcal{T}^{MAX}} (1 - c)^T & \text{if } \mu_0 < 1/2.
\end{cases}
\]

It will turn out that for \( \gamma \) large enough (i.e, taking action 1 is more valuable for the designer), the optimal design is almost-geometric with terminal time \( T < \mathcal{T}^{MAX} \) since doing so increases the probability that the DM takes action 1, albeit at an earlier time. This logic is reversed for \( \gamma \) small.

**Qualitative feature of belief paths.** For the case in which \( \mu_0 \geq 1/2 \), the DM’s prior is already aligned with the designer’s preference. In this case, we see that beliefs move up gradually. For the case in which \( \mu_0 < 1/2 \), however, the DM’s prior is misaligned with the designer’s interest. We see that almost-geometric structures can be understood in three stages:

**Figure 3:** Illustration of continuation beliefs under almost-geometric information structures

(i) **Initial jump** in beliefs: \( \mu_1 = \frac{\mu_0}{2\mu_0 - c} > 1/2. \)
(ii) Gradual increase for times $1 \leq t \leq T$: $\mu_{t+1} = \frac{\mu_t}{1-c}$ which is approximately “smooth” for small $c$.

(iii) Persuasion period at which the information structure terminates at $T$: the DM is thus left with belief $\mu_T > 1/2$ so that conditional on staying until time $T$, she takes action $1$.

Figure 3 shows the path of continuation beliefs for almost-geometric information structures. For $\gamma \geq 99$, single-period persuasion is approximately optimal; for $\gamma = 50$, the optimal dynamic information structure exhibits bad news at a rate so that beliefs initially jump to $\approx 0.5$ at which the DM is approximately indifferent between action $0$ and $1$. For times $1 < t \leq 49$, bad news then arrives at a constant conditional rate $c$ which keeps DM indifferent between stopping and continuing. At the terminal time $T = 49$, the DM then takes action $1$.

4 Concluding Remarks

We have shown that for binary choice problems where waiting is costly, the set of feasible joint distributions over actions and stopping times is pinned down by a set of finite linear inequalities. The key were the observations that (i) it suffices to consider a single continuation belief; and (ii) any distribution of actions can always be implemented through extremal beliefs. We then employed this characterization to solve a series of information design problems in which the designer has preferences over both DM’s actions and stopping times.

References


Appendix to ‘Feasible Joint Distributions over Actions and Stopping Times’
Andrew Koh and Sivakorn Sanguanmoo

Appendix A: Proofs of Optimal Design Results

A.1 Preliminaries. The proofs in this section rely on a belief-based approach to the information structure. From Proposition 1 (proved below) we know that it is without loss to consider stopping beliefs in the set \{0, \hat{\mu}, 1\}. Furthermore, from Proposition 1, we know that it is without loss to consider information structures with a unique continuation belief. Finally, we break ties in favor of action 1 so that the sender’s optimal is attainable which implies that joint distributions \(\Delta(A \times \mathcal{T})\) can be written as distributions \(\Delta(\{0, \hat{\mu}, 1\} \times \mathcal{T})\). These three observations imply that we can work directly on joint distributions over stopping beliefs and stopping times. Extremal and obedient dynamic information structures can thus be summarized by a sequence of probabilities \(p := (p_{t,i})_{p \in \{0,0,1\}, t \in \mathcal{T}} \in \Delta(\{0, \hat{\mu}, 1\} \times \mathcal{T})\), noting that this also pins down the unique belief path: if the DM has paid attention up to time \(t\), her beliefs at \(t\) are, by Bayes’ rule

\[
\mu_t^C(p) := \mathbb{P}_t^p(\theta = 1|\tau > t) = \frac{\sum_{s=t+1}^{\infty} \sum_{\mu^s \in \{0,\hat{\mu},1\}} \mu^s p_{t+1}}{\sum_{s=t+1}^{\infty} \sum_{\mu \in \{0,\hat{\mu},1\}} \mu \cdot p_s}
\]

where we sometimes drop the dependence on \(p\) when there is no ambiguity about the dynamic information structure it is with respect to. The next lemma gives conditions under which \(p\) is feasible:

**Lemma 1.** A distribution \(p\) over \(\Delta(\{0, \hat{\mu}, 1\} \times \mathcal{T})\) is feasible if and only if the following constraints hold:

(i) (Martingale constraint) \(\mu_0 = \sum_{t \in \mathcal{T}} \sum_{\mu^s \in \{0,\hat{\mu},1\}} \mu^s p_{s,m}\).

(ii) (Obedience constraint) For every \(t \in \mathcal{T},\)

\[
\sum_{s=t+1}^{\infty} \sum_{\mu \in \{0,\hat{\mu},1\}} \mu \cdot p_{s,m} \left( u^*(\mu_t^C) - c(t) \right) \leq \sum_{s=t+1}^{\infty} \sum_{\mu^s \in \{0,\hat{\mu},1\}} \mu^s \cdot p_{s,m} \left( u^*(\mu^s) - c(s) \right),
\]

where we write \(u^*(\theta) := \max_a \mathbb{E}^\mu[u(a, \theta)]\) to denote the expect utility from the decision problem under belief \(\mu\).

It will be convenient to write the LHS of the obedient constraint as follows:

\[
\max_{a \in \{0,1\}} \left\{ \left( \sum_{s=t+1}^{\infty} \sum_{\mu^s \in \{0,\hat{\mu},1\}} m \cdot p_{s,m} \right) u(a, 1) + \left( \sum_{s=t+1}^{\infty} \sum_{\mu \in \{0,\hat{\mu},1\}} (1 - m) \cdot p_{s,m} \right) u(a, 0) \right\} - \left( \sum_{s=t+1}^{\infty} \sum_{\mu \in \{0,\hat{\mu},1\}} p_{s,m} \right) c(t).
\]

This means the obedient constraint for each time \(t \in \mathcal{T}\) consists of two linear constraints (when \(a = 0\) and \(a = 1\), which implies we can write down the sender’s optimization problem with all linear constraints in variables \(\{p_{s,m}\}_{m \in \{0,\hat{\mu},1\}, t \in \mathcal{T}}\). Denote \(\mathcal{P}^*\) as the set of feasible distributions over \(\Delta(\{0, \hat{\mu}, 1\} \times \mathcal{T})\) that solve the sender’s optimization problem.

Electronic copy available at: https://ssrn.com/abstract=4595819
Finally, we normalize \( u(1, 0) = u(0, 1) = 0 \) without loss.

A.2 Proof of Proposition 2 (Optimal structures when \( v \) additively separable).

Proof. Additive separable implies supermodular so applying Proposition 3, if \( \mathcal{P}^*_{\text{full}} \neq \emptyset \), then a sender’s optimal information structure can be obtained by using full information signals only, which means that the DM’s expected action coincides with her prior beliefs. By additive separability, it is sufficient for the sender to just consider the stopping time, which falls into the pure attention case. Next consider the case in which \( \mathcal{P}^*_{\text{full}} = \emptyset \) and Lemma 3 implies \( \mathcal{P}^*_{\text{bad}} \neq \emptyset \). For any \( p^* \in \mathcal{P}^*_{\text{bad}} \), suppose \( p^* \) has a terminal time \( T \). Let \( t_0 \) be the last time the information structure provides a stopping message with positive probability before time \( T \). We will show that the sender’s utility does not change under a suitable modification so that the information structure ends at time \( t_0 \) instead.

At time \( t_0 \), the information structure sends either stopping message 0 or continuation message at which the DM is indifferent between stopping or continuing paying attention. This means the continuation belief at time \( t \) must be strictly more than \( \bar{\mu} \); otherwise there is no net value of further information. For each \( \mu^S = \{0, \bar{\mu}, 1\} \), let

\[
P_{\mu^S} := \mathbb{P}(m_T = m_{\mu^S} | r > t_0)
\]

be the conditional probability of receiving message \( m_{\mu^S} \) at time \( T \) given that the DM pays attention until time \( t_0 \). We can write down the system of equations for \( P_0, P_1, \) and \( P_{\bar{\mu}} \) as follows:

\[
\begin{align*}
1 &= P_0 + P_1 + P_{\bar{\mu}} \quad \text{(Total probability)} \\
\mu_{t_0}^C &= P_1 + \bar{\mu} P_{\bar{\mu}} \quad \text{(Martingale)} \\
u^*(\mu_{t_0}^C) &= \mu_{t_0}^C u(1, 1) = P_0 u^*(0) + P_1 u^*(1) + P_{\bar{\mu}} u^*(\bar{\mu}) - c(T - t_0). \quad \text{(Obedience)}
\end{align*}
\]

This implies \( P_0 = c(T - t_0)/u(0, 0) \). If the designer stops providing information at time \( t_0 \) and lets the DM takes action 1 right away, the designer’s expected utility after time \( t_0 \) is \( h(t_0) + \mu_{t_0} \). On the other hand, under \( p^* \), the sender’s expected utility after time \( t_0 \) is \( h(T) + [1 - c(T - t_0)/u(0, 0)] \) where the second term is the probability of the DM taking action 1 at \( T \). By the optimality of \( p^* \), we must have

\[
h(t_0) + \mu_{t_0} \leq h(T) + [1 - c(T - t_0)/u(0, 0)].
\]

Modify the information structure as follows: let \( p' \) be a distribution over \( \Delta(\{0, \bar{\mu}, 1\} \times \mathcal{T}) \) such that

\[
\begin{align*}
p'_{\mu, t_0} &= \epsilon \\
p'_0, t_0 &= p_0, t_0 - \epsilon \\
p'_{\mu, T} &= p_{\mu, T} - \epsilon \\
p'_0, T &= p_{0, T} + \epsilon,
\end{align*}
\]
Lemma 2. Following lemma. We will show that continuing upon receipt of the message the DM stopping before time (2011).

\[ \text{STOP}_{s}^{b'} - \text{STOP}_{s}^{p^*} = -u(1,1)\bar{\mu} \epsilon < 0, \]
\[ \text{CON}_{s}^{b'} - \text{CON}_{s}^{p^*} = u(0,0)(1 - \bar{\mu})\epsilon. \]

Since \( p^* \) is feasible, \( \text{STOP}_{s}^{p^*} \leq \text{CON}_{s}^{p^*} \) which thus implies \( \text{STOP}_{s}^{b'} \leq \text{CON}_{s}^{b'} \) so that the obedience constraint is fulfilled for every time \( s \). Thus \( p' \) is a feasible distribution. Now observe that the designer’s value function is additively separable hence both \( p' \) and \( p^* \) yields the same expected value. Thus, \( p^* \in \mathcal{P}^* \). Under \( p' \), the sender stops providing information when he sends message \( \bar{\mu} \) at time \( t_0 \). This means the sender’s utility for providing more information at node \( \bar{\mu} \) by persuading at time \( T \) is worse than that for stopping right away. With the same calculation as before, the sender’s incentive to stop providing information implies
\[ h(t_0) + \bar{\mu} \geq h(T) + [1 - c(T - t_0)/u(0,0)] \]

With the first equation and since \( \mu_{h} \geq \bar{\mu} \), the above inequality is in fact an equality. Therefore, the sender’s utility does not change when he ends the game at time \( t_0 \), as desired. Hence we can shorten the information structure without hurting the sender if there exists a stopping signal before the last period. By recursively shortening the information structure, we finally obtain the sender’s optimal information structure such that the designer only gives the DM a stopping message with positive probability at a single terminal time. □

A.3 Proof of Proposition 3 (Optimal structures when \( v \) supermodular). Before we prove Proposition 3 we develop some notation. Let \( \mathcal{P}^* \) be the set of feasible distributions which solve the sender’s problem. Define

\[ \mathcal{P}_{\text{full}}^* := \{ p \in \mathcal{P}^* : \forall t \in \mathcal{T}, p_{\bar{\mu},t} = 0 \} \]
\[ \mathcal{P}_{\text{ter}}^* := \{ p \in \mathcal{P}^* : \text{there exists a unique } T > 0 \text{ such that } p_{\bar{\mu},T} > 0 \} \]

\[ \mathcal{P}_{\text{bad}}^* := \{ p \in \mathcal{P}^* : \begin{array}{l}
(\text{i}) \text{ there exists a unique } T > 0 \text{ such that } p_{\bar{\mu},T} > 0, \\
(\text{ii}) \ p_{\bar{1},s} = 0 \text{ for all } s < T, \text{ and } \\
(\text{iii}) \ \text{For every } t < T \text{ such that } p_{0,t} > 0, \text{ DM is indifferent between stopping and continuing at time } t \text{ upon receipt of } m_0. \end{array} \} \]

\( \mathcal{P}_{\text{full}}^* \) (“full information”) corresponds to distributions under which the DM only stops when she obtains full information. \( \mathcal{P}_{\text{ter}}^* \) (“terminal”) corresponds to distributions under which there is a fixed terminal time \( T \) when the DM could stop with belief \( \bar{\mu} \); at this terminal time, the solution is similar to the single-period Bayesian persuasion solution Kamenica and Gentzkow (2011). \( \mathcal{P}_{\text{bad}}^* \) (“bad news”) corresponds to distributions which also persuade at some terminal time \( T \) but in addition, the DM only receives bad news for all times \( t < T \) i.e., conditional on stopping before time \( t \), the DM is certain that \( \theta = 0 \) and furthermore, on times \( t < T \) where the DM could receive bad news (i.e., \( p_{0,t} > 0 \)), the DM is indifferent between stopping and continuing upon receipt of the message \( m_0 \). Clearly \( \mathcal{P}_{\text{bad}}^* \subseteq \mathcal{P}_{\text{full}}^* \).

We will show that \( \mathcal{P}_{\text{full}}^* \cup \mathcal{P}_{\text{bad}}^* \neq \emptyset \) which is equivalent to Proposition 3. We begin with the following lemma.

Lemma 2. \( \mathcal{P}_{\text{full}}^* \cup \mathcal{P}_{\text{bad}}^* \neq \emptyset \)

Electronic copy available at: https://ssrn.com/abstract=4595819
Proof of Lemma 2. By Corollary 1 in Appendix C, the optimization problem $\min_{p \in P^*} \sum_{t=1}^{\infty} 2^{-t} p_{\mu,t}$ has a solution. Suppose that $p^* \in P^*$ solves such an optimization problem. We will show that $p^*$ must satisfy either condition in Proposition 1. If $p^*$ does not satisfy the first condition, there exists $t \in T$ such that $p_{\mu,t} > 0$.

Define $T := \min\{t \in T : p_{\mu,t} > 0\}$ as the first time that under the information structure $p^*$, the DM stops with belief $\mu$, and $T_{\mu} := \min\{t \geq T : p_{\mu,t} > 0\}$ for $\mu \in \{0, 1\}$ as the times after $T_\mu$ under which the information structure leaves the DM with stopping belief either 0 or 1. Note that if the information structure does not end at time $T_\mu$, this implies $\{t > T : p_{\mu,t} > 0\}$ and $\{t > T : p_{0,t} > 0\}$ are both non-empty. To see this, observe that otherwise the DM takes the same action for every stopping time $t > T$, which cannot incentivize her to pay attention at time $T_\mu$. As such, it will suffice to show that we cannot have $T_0 > T$ or $T_1 > T$. We proceed by considering cases.

(C1) $\mu_T^{p^*} < \bar{\mu}$. Suppose towards a contradiction that $T_0 > T$. Modify the information structure as follows: let $p'$ be a distribution over $\Delta(\{0, \bar{\mu}, 1\} \times T)$ such that, with small $\epsilon > 0$,

$$p'_{0,T} = p_{0,T} + \epsilon,$$

$$p'_{\bar{\mu},T} = p_{\bar{\mu},T} - \epsilon,$$

$$p'_{0,T_0} = p_{0,T_0} - \epsilon,$$

$$p'_{\mu,T_0} = p_{\mu,T_0} + \epsilon,$$

and $p'_{a,t} = p_{a,t}^*$ for every other pair of $(a,t)$. It is easy to see that $p'$ is a well-defined probability distribution and satisfies a martingale condition. We will show that obedient constraints hold under $p'$. Obedient constraints clearly do not change at time $t < T$ and at time $t \geq T_0$ as continuation beliefs and values do not change. Consider any time $T \leq t < T_0$. By the definition of $T_0$, $p'_{0,s} = p'_{0,t} = 0$ for any $T \leq s < t$. The martingale condition implies that $\mu_T^{p'}$ is a positive linear combination of $\mu_T^{p^*}$ and stopping beliefs between time $T$ and $s$ which are either 1 or $\bar{\mu}$. Since $\mu_T^{p'} < \bar{\mu}$, $\mu_T^{p^*}$ must also be less than $\bar{\mu}$. Similarly, we also get $\mu_s^{p'} < \bar{\mu}$. This means action 0 is the optimal action at time $s$ under both $p$ and $p'$. Now observe that since the receiver would take action 0 if she stops at time $s$, we have

$$\mu_s^{p'}(p') - \mu_s^{p^*}(p^*) = \epsilon\bar{\mu} \implies STOP_s^{p'} - STOP_s^{p^*} = -u(0,0)\epsilon.$$ 

Furthermore, we also have

$$\text{CONT}_s^{p'} - \text{CONT}_s^{p^*} = (p'_{\mu,T_0} - p_{\mu,T_0})\epsilon(u(0,1) - u(0,0)) = -u(0,0)\epsilon.$$ 

Since $p^*$ is a feasible distribution, $STOP_s^{p^*} \leq \text{CONT}_s^{p^*}$ which also implies $STOP_s^{p'} \leq \text{CONT}_s^{p'}$ hence the obedient constraint at time $T \leq s < T_0$ holds under $p'$. Thus, $p'$ is a feasible distribution.
Now observe $p' \in \mathcal{P}^*$ because

$$\mathbb{E}^{p'}[f(a, \tau)] - \mathbb{E}^{p*}[f(a, \tau)] = \left[ (f(1, T_0) - f(0, T_0)) - (f(1, T) - f(0, T)) \right] \epsilon \geq 0$$

by the supermodularity of $f$. However,

$$\sum_{t=1}^{\infty} 2^{-t} p'_{t,t} - \sum_{t=1}^{\infty} 2^{-t} p^*_{t,t} = (2^{-T_0} - 2^{-T}) \epsilon < 0,$$

which contradicts the fact that $p^*$ solves the optimization problem $\min_{p \in \mathcal{P}^*} \sum_{t=1}^{\infty} 2^{-t} p_{t,t}$.

(C2) $\mu_T^{p'} > \mu$. Suppose towards a contradiction that $T_1 > T$. We modify the information structure as follows: let $p'$ be a distribution over $\Delta(\{0, \mu, 1\} \times \mathcal{T})$ such that, with small $\epsilon > 0$,

$$p'_{1,t} = p^*_{1,t} + \epsilon,$$
$$p'_{1,t} = p^*_{1,t} - \epsilon,$$
$$p'_{1,t_1} = p^*_{1,t_1} + \epsilon,$$
$$p'_{1,t_1} = p^*_{1,t_1} - \epsilon,$$
$$p'_{1,t} = p^*_{1,t} + \epsilon,$$
$$p'_{1,t} = p^*_{1,t} - \epsilon$$

and $p'_{a,t} = p^*_{a,t}$ for every other pair of $(a, t)$. It is easy to see that $p'$ is a well-defined probability distribution and satisfies a martingale condition. We will show that obedient constraints hold under $p'$. Obedient constraints clearly do not change at time $t < T$ and time $t \geq T_1$ as continuation beliefs and values do not change. Consider any time $T \leq t < T_0$. By the definition of $T_0$, $p^*_{1,s} = p'_{1,s} = 0$ for any $T \leq s < t$. The martingale condition implies that $\mu_T^{p'}$ is a positive linear combination of $\mu_T^{p'}$ and stopping beliefs between time $T$ and $s$ which are either 0 or $\tilde{\mu}$. Since $\mu_T^{p'} > \tilde{\mu}$, $\mu_T^{p'}$ must also be greater than $\tilde{\mu}$. Similarly, we also get $\mu_T^{p'} > \mu$. This means both $p^*$ and $p'$ yield the same optimal action 1 at time $s$. Now under the modification to $p^*$ we have that

$$\mu^{p'}_s(p') - \mu^{p'}_s(p^*) = (1 - \tilde{\mu}) \epsilon \implies STOP_s^{p'} - STOP_s^{p} = -(1 - \tilde{\mu}) u(1, 1) \epsilon.$$

Furthermore, we have

$$CON_s^{p'} - CON_s^{p} = -(1 - \tilde{\mu}) u(1, 1) \epsilon.$$

Since $p^*$ is a feasible distribution, $STOP_s^{p} \leq CON_s^{p}$ which implies $STOP_s^{p'} \leq CON_s^{p'}$. This means the obedient constraint at time $s$ holds under $p'$ at any time $T \leq s < t_0$. Thus, $p'$ is a feasible distribution. Note that $p' \in \mathcal{P}^*$ since both $p^*$ and $p'$ yield the same joint distribution over actions and times. However,

$$\sum_{t=1}^{\infty} 2^{-t} p'_{t,t} - \sum_{t=1}^{\infty} 2^{-t} p^*_{t,t} = (2^{-T_0} - 2^{-T}) \epsilon < 0,$$
which contradicts the fact that $p^*$ solves the optimization problem $\min_{p \in \mathcal{P}^*} \sum_{t=1}^{\infty} 2^{-t} p_{\mu,t}$.

(C3) $\mu^p_t = \bar{\mu}$. We modify the information structure as follows: let $p'$ be a distribution over $\Delta(\{0, \bar{\mu}, 1\} \times T)$ such that, with $\epsilon$ in a small neighborhood of 0,

$$p'_{\mu,t} = p^*_{\mu,t} - \sum_{m',t'=T} p^*_{m',t'} \epsilon,$$

$$p'_{m,t} = p^*_{m,t} + p^*_{m,t} \epsilon \quad \text{for all} \ m \in \{0, \bar{\mu}, 1\}, \ t > T,$$

and $p'_{a,t} = p^*_{a,t}$ for every other pair of $(a, t)$. Intuitively speaking, we decrease the probability of stopping at belief $\bar{\mu}$ at time $T$ but increase the continuation probability at time $T$ instead. It is easy to see that $p'$ is a well-defined probability distribution and satisfies a martingale condition. We will show that the obedient constraints hold under $p'$. Obedient constraints hold at time $t > T$ because

$$\text{STOP}_{t}' = (1 + \epsilon)\text{STOP}_{t}^p \leq (1 + \epsilon)\text{CONT}_{t}^p = \text{CONT}_{t}'^p.$$

For $t < T$, $\text{STOP}_{t}' = \text{STOP}_{t}^p$ because the continuation belief and probability do not change at time $t$. Moreover,

$$\text{CONT}_{t}^p - \text{CONT}_{t}'^p = \sum_{s > T} \sum_{m \in \{0, \bar{\mu}, 1\}} (v^*(m) - c(s))p^*_{m,s} - \sum_{m',t'>T} p^*_{m',t'}(v^*(\bar{\mu}) - c(T)) \geq 0,$$

where the inequality follows by the obedient constraint at time $T$. Therefore, $p$ is a feasible distribution. Note that $U_S^p - U_S^p$ is linear in $\epsilon$. Since $p' \in \mathcal{P}^*$, we have $\mathbb{E}^p[f(a, \tau)] = \mathbb{E}^p[f(a, \tau)]$ for every $\epsilon$ in a small neighborhood around 0. But taking a small enough $\epsilon > 0$,

$$\sum_{t=1}^{\infty} 2^{-t} p'_{\mu,t} - \sum_{t=1}^{\infty} 2^{-t} p^*_{\mu,t} = \sum_{m,t>T} (2^{-t} - 2^{-T}) p^*_{m,t} \epsilon < 0,$$

which contradicts the fact that $p^*$ solves the optimization problem $\min_{p \in \mathcal{P}^*} \sum_{t=1}^{\infty} 2^{-t} p_{\mu,t}$.

We now state and prove a lemma which will be useful to complete the proof.

**Lemma 3.** Suppose $p^* \in \mathcal{P}^{**}$. If $U$ is strictly supermodular, then for every time $t$ such that $\mu^C_t(p^*) < \bar{\mu}$ we have $p^*_{1,t} = 0$.

**Proof.** Suppose, towards a contradiction that there exists time $t$ such that $p^*_{1,t} \neq 0$ and $\mu^p_t < \bar{\mu}$. Let $T_0 = \min\{s > t : p^*_{0,s} > 0\}$. We modify the information structure as follows: let $p'$ be a distribution over $\Delta(\{0, \bar{\mu}, 1\} \times T)$ such that, with small $\epsilon > 0$,
and \( p'_{m,t} = p'_{m,t} \) for all other pairs of \((m,t)\). It is easy to see that \( p' \) is a well-defined probability distribution and satisfies the martingale condition. We will show that obedient constraints hold under \( p' \). Obedient constraints clearly do not change at times \( s < t \) and times \( s \geq T_0 \) as continuation beliefs and values do not change.

Now consider any time \( s \) where \( t \leq s < T_0 \). The martingale condition implies that \( \mu^p_s \) is a positive linear combination of \( \mu^p_t \) and stopping beliefs between time \( t \) and \( s \) which are always 1 by the definition of \( T_0 \) and Lemma 2. Since \( \mu^p_t < \bar{\mu} \), \( \mu^p_s \) must also be less than \( \bar{\mu} \). Similarly, we also get \( \mu^p_s < \bar{\mu} \). This means action 0 is the optimal action at time \( s \) under both \( p^* \) and \( p' \). This implies

\[
\mu^C_s(p') - \mu^C_s(p^*) = -\epsilon \implies \text{STOP}^p_s - \text{STOP}^s_s = -\epsilon u(0, 0)
\]

and furthermore,

\[
\text{CONT}^p_s - \text{CONT}^s_s = (u(1, 1) - u(0, 0))\epsilon.
\]

Putting these inequalities together we then have

\[
\text{CONT}^p_s - \text{STOP}^p_s > \text{CONT}^s_s - \text{STOP}^s_s.
\]

and since \( p^* \) is feasible, the obedient constraint at time \( s \) holds under \( p' \) so it is feasible too. Finally note that under \( p' \) we have

\[
\mathbb{E}^p[f(a, \tau)] - \mathbb{E}^p[f(a, \tau)] = (f(1, T_0) - f(0, T_0)) - (f(1, t) - f(0, t))\epsilon > 0
\]

by the strict supermodularity of \( U \), which contradicts the optimality of \( p^* \).

\[\square\]

**Lemma 4.** \( \mathcal{P}^*_\text{full} \cup \mathcal{P}^*_\text{bad} \neq \emptyset \)

**Proof.** If \( \mathcal{P}^*_\text{full} \neq \emptyset \), there is nothing to show. Next suppose that \( \mathcal{P}^*_\text{full} = \emptyset \). Lemma 2 implies \( \mathcal{P}^*_\text{per} \neq \emptyset \). Pick \( p^* \in \mathcal{P}^*_\text{per} \) and let \( T \) be the terminal time of \( p^* \).

First, we prove the case in which \( f \) is strictly supermodular. Since \( p^* \) is a sender’s optimal
information structure, it must solve the following optimization problem

\[
\max_{\{p_{0,t}, p_{1,t} \}_{t=1}^T, p_{\bar{\mu}, T}} \mathbb{E}^p [f(a, \tau)] = \sum_{t=1}^T p_{0,t} f(0, t) + \sum_{t=1}^T p_{1,t} f(1, t) + p_{\bar{\mu}, T} f(1, T)
\]

s.t. \[ \sum_{t=1}^T p_{0,t} + \sum_{t=1}^T p_{1,t} + p_{\bar{\mu}, T} = 1 \] (Total probability)

\[ \sum_{t=1}^T p_{1,t} + \bar{\mu} p_{\bar{\mu}, T} = \mu \] (Martingale)

\[ \nu^*(\mu) \leq \sum_{t=1}^T p_{0,t}(\nu^*(0) - ct) + \sum_{t=1}^T p_{1,t} U(\nu^*(1) - ct) + p_{\bar{\mu}, T}(\bar{\nu} - cT) \] (Obedience-0)

Obedience-(a, t) for all \(a \in \{0, 1\}\) and all \(t \in \{1, \ldots, T - 1\}\) (Obedience-(a,t))

\[ p_{0,t} \geq 0, p_{1,t} \geq 0, p_{\bar{\mu}, T} \geq 0 \text{ for all } t \in \{1, \ldots, T\} \] (Non-negativity)

Note that all constraints of this optimization problem are linear, and the optimization problem is linear. Define \(\mathcal{R}\) as the polytope of \(\{(p_{0,t}, p_{1,t})_{t=1}^T, p_{\bar{\mu}, T}\}\) which satisfies the above constraints. By standard arguments, a solution to the above optimization problem lies in the extreme points of \(\mathcal{R}\), which we denote with Ext \(\mathcal{R}\). This implies Ext \(\mathcal{R} \cap \mathcal{P}^* \neq \emptyset\) and consider any \(p^* \in \text{Ext } \mathcal{R} \cap \mathcal{P}^*\).

**Step 1:** \(p_{0,T}^*, p_{1,T}^*, p_{\bar{\mu}, T}^* > 0\).

If \(p_{\bar{\mu}, T}^* = 0\), then \(p^* \in \mathcal{P}^*_{\text{full}}\), a contradiction. Therefore, \(p_{\bar{\mu}, T}^* > 0\). If \(p_{0,T}^* = 0\), then the DM takes the same action 1 at time \(T\), so there is no value of the information at time \(T\) which violates the obedience constraint at time \(T - 1\). Therefore, \(p_{0,T}^* > 0\). Similarly, if \(p_{1,T}^* = 0\) then taking action 0 is always an optimal action at time \(T\) (even though she breaks ties in favor of 1). This once again violates the obedience constraint at time \(T - 1\).

**Step 2:** For each \(1 \leq t < T\), at most two of the following four constraints bind: Obedience-(0, t), Obedience-(1, t), \(p_{0,t}^* \geq 0\), \(p_{1,t}^* \geq 0\)

Suppose that both Obedience-(0, t) and Obedience-(1, t) bind. This means the utilities of taking action 0 and 1 at time \(t\) are the same, which implies \(\mu_t = \bar{\mu}\), and the DM must be indifferent between continuing and stopping at time \(t\). If \(p_{0,t}^* = 0\), then the DM’s utility after time \(T\) is equivalent to that when she always takes action 1. Thus, there is no value of information after time \(t\), which contradicts the obedience constraint at time \(t - 1\). Therefore, \(p_{0,t}^* > 0\), and, similarly, \(p_{1,t}^* > 0\). This means only two constraints bind, as desired. Suppose that both \(p_{0,t}^* = 1\) and \(p_{1,t}^* = 0\). This means the sender does not send any signal at time \(t\). Then any obedience constraint at time \(t\) cannot bind; otherwise, the obedience constraint at time \(t - 1\) would be violated, as desired.

**Step 3:** For each \(1 \leq t < T\) Obedience-0 binds and exactly two of the following four constraints bind: (i) Obedience-(0, t); (ii) Obedience-(1, t); (iii) \(p_{0,t} \geq 0\); and (iv) \(p_{1,t} \geq 0\)

Noting that the optimization problem had \(2T + 1\) variables, then since \(p^* \in \text{Ext } \mathcal{R}\), \(2T + 1\) constraints must bind (Simon, 2011, Proposition 15.2). From Step 1 we know that none of the non-negativity constraints at time \(T\) bind. From Step 2, we know that for each \(1 \leq t < T\)
we have obedience constraints fulfilled for all times so

Since we have

which holds no matter whether the DM optimally takes action

implies

is a feasible distribution,

Note that part (i) or Step 4 is exactly equal to condition (iii) of \( \mathcal{P}^*_{\text{bad}} \) so it remains to show that for all \( t < T \), \( p^*_{1,t} = 0 \) so that we have a pure bad news dynamic information structure (condition (ii) of \( \mathcal{P}^*_{\text{bad}} \)).

**Step 5: truncate \( p^* \).**

If \( p^*_{1,t} = 0 \) for every \( t < T \), then \( p^* \in \mathcal{P}^*_{\text{bad}} \) as desired. Suppose that \( p^*_{1,t} > 0 \) for some \( t < T \).

Define \( T_0 := \min\{ t \leq T : p^*_{1,t} > 0 \} \). Step 4 implies \( \mu^*_{1,t} = \bar{\mu} \). We modify the information structure as follows: let \( p' \) be a distribution over \( \Delta(\{0, \bar{\mu}, 1\} \times T) \) such that

\[
\begin{align*}
p'_{\mu,T_0} &= \epsilon \\
p'_{1,T_0} &= p^*_{1,t} - \epsilon \\
p'_{\mu,T} &= p^*_{\mu,T} - \epsilon \\
p'_{1,T} &= p^*_{1,t} + \epsilon,
\end{align*}
\]

and \( p'_{m,t} = p^*_{m,t} \) for all other \((m, t)\) pairs. It is easy to check that \( p' \) is a probability distribution and satisfies the martingale condition, and that obedience constraints do not change at time \( s < T_0 \). When \( s > T_0 \), observe

\[
\mu^C(p')(1 - \bar{\mu}) = \mu^C(p^*) \quad \Rightarrow \quad \text{STOP}_s^{p'} - \text{STOP}_s^{p^*} \leq u(1, 1)(1 - \bar{\mu})
\]

which holds no matter whether the DM optimally takes action 0 or 1 if she stops at \( s \). Furthermore, we have

\[
\text{CONT}_s^{p'} - \text{CONT}_s^{p^*} = u(1, 1)(1 - \bar{\mu})
\]

Since \( p^* \) is a feasible distribution, \( \text{STOP}_s^{p^*} \leq \text{CONT}_s^{p^*} \). Therefore, \( \text{STOP}_s^{p'} \leq \text{CONT}_s^{p'} \) hence we have obedience constraints fulfilled for all times so \( p' \) is a feasible distribution. Note that
Because both \( p^{**} \) and \( p' \) yield the same joint distribution of actions and times (since we break DM indifference in favour of action 1).

Now observe that under \( p^{**} \), when the DM is indifferent at time \( T_0 \) with belief \( \bar{\mu} \) (from Step 4), the designer provides more information until time \( T \). On the other hand, under \( p' \), when the DM reaches the stopping belief \( \bar{\mu} \) at time \( T_0 \), the sender prefers to stop providing further information. To see this, assume, towards a contradiction, that this is not the case so that the sender strictly prefers to provide further information at time \( T_0 \) when the continuation belief is \( \bar{\mu} \). We have shown that under \( p' \), the designer obtains the same utility and furthermore, \( p'_{\bar{\mu},T_0} > 0 \). But since the sender strictly prefers to continue at time \( T_0 \), it cannot find it optimal to have the DM stop at time \( T_0 \) with belief \( \bar{\mu} \) which contradicts the optimality of \( p' \) and hence \( p^{**} \).

Truncate \( p^{**} \) at time \( T_0 \) by constructing \( p^{***} \) as follows:

\[
P^{***}_{m,t} := \begin{cases} 
P^{**}_{m,t} & t < t_0 \\
P^{**}_{m,t} & t = t_0, m \in \{0,1\} \\
\sum_{s > t_0} \sum_m p^{**}_{m,s} & t = t_0, m = \bar{\mu} \\
0 & t > t_0,
\end{cases}
\]

and by the previous argument, \( p^{***} \) and \( p^{**} \) yield the same sender’s utility. Furthermore, Thus, \( p^{***} \in \mathcal{P}^* \). From the definition of \( t_0 \), it is simple to show that \( p^{***} \in \mathcal{P}^*_{bad} \), as desired.

Now suppose that \( f \) is supermodular but not necessarily strictly supermodular. Define \( f^\varepsilon : \mathcal{A} \times \mathcal{T} \rightarrow \mathbb{R} \) such that \( f^\varepsilon (1,t) = f (1,t) + \varepsilon t \) and \( f^\varepsilon (0,t) = f (0,t) \) for every \( \varepsilon > 0 \). Clearly \( f^\varepsilon \) is strictly supermodular. For each \( \varepsilon > 0 \), pick \( p^\varepsilon \in \mathcal{P}^*_{\text{bad}} (f^\varepsilon ) \cup \mathcal{P}^*_{\text{full}} (f^\varepsilon ) \neq \emptyset \) where we know that this is non-empty from the previous case.\(^6\) Because \( \mathcal{D} \subseteq \Delta (\Delta (\Theta \times \mathcal{T})) \), the space of feasible joint distributions over beliefs and stopping times is compact under the metric \( \Delta \), this means there exists a sequence of \( \{\varepsilon_n\} \subset \mathbb{R}^+ \) such that \( \lim_{n \to \infty} \varepsilon_n = 0 \) and \( \lim_{n \to \infty} p^{\varepsilon_n} =: \bar{p}^* \) under the metric \( \Delta \). From Berge’s theorem of the maximum, the designer’s optimum under \( f^\varepsilon \) is continuous in \( \varepsilon \) so \( \bar{p}^* \) yields an optimal solution under the sender’s function \( f \). It remains to show that \( \bar{p}^* \in \mathcal{P}^*_{\text{bad}} (f) \cup \mathcal{P}^*_{\text{full}} (f) \).

If \( \bar{p}^*_{\mu,t} = 0 \) for every \( t \in \mathcal{T} \), then \( \bar{p}^* \in \mathcal{P}^*_{\text{full}} \) as desired. Now suppose that there exists \( T_0 \in \mathcal{T} \) such that \( \bar{p}^*_{\mu,T_0} > 0 \). This means \( \bar{p}^{\varepsilon_n}_{\mu,T_0} > 0 \) for large \( n > N \), which implies \( p^{\varepsilon_n} \in \mathcal{P}^*_{\text{bad}} (U^\varepsilon) \). Thus, for every \( n > N \), we must have

(i) \( p^{\varepsilon_n} \) ends at time \( T_0 \),

(ii) For every \( t < T_0 \), \( p^{\varepsilon_n}_{1,t} = 0 \).

(iii) For every \( t < T_0 \) such that \( \bar{p}^{\varepsilon_n}_{0,t} > 0 \), the DM must be indifferent between stopping or continuing to pay attention at time \( t \) under \( p^{\varepsilon_n} \).

It is clear that \( \bar{p}^*_{1,t} = \lim_{n \to \infty} p^{\varepsilon_n}_{1,t} = 0 \) for every \( t < T_0 \). Consider \( t < T_0 \) such that \( \bar{p}^*_{0,t} > 0 \). This means there exists a subsequence \( \{\varepsilon_{n_i}\} \) such that \( p^{\varepsilon_{n_i}}_{0,t} > 0 \) for every \( i \). Therefore, the DM must be indifferent between stopping or continuing to pay attention at time \( t \) under \( p^{\varepsilon_{n_i}} \). for every \( i \).

\(^6\) We use the notation \( \mathcal{P}^* (f)_{\text{full}} \) to denote the class of optimal information structures under which stopping beliefs are only 1 and 0 when the designer’s value function if \( f \), and analogously for \( \mathcal{P}^*_{\text{bad}} \).

converge to $STOP_t^{\bar{\rho}^*}$ and $CONT_t^{\bar{\rho}^*}$, respectively. This implies $STOP_t^{\bar{\rho}^*} = CONT_t^{\bar{\rho}^*}$, which means the DM must be indifferent between stopping or continuing to pay attention at time $t$ under $\bar{\rho}^*$. Therefore, $\bar{\rho}^* \in \mathcal{P}^*_\text{bad}(f)$, as desired.

A.4 Calculations Supporting Example [[X]]. Suppose that the attention function is linear: $v(t) = t$ and $\frac{\phi(\mu_t)}{c}$ is integer. Under the pure attention case, the sender simply reveals full information at time $\frac{\phi(\mu_t)}{c}$ and obtains utility $\frac{\phi(\mu_t)}{c} + \lambda \mu_0$. Under the pure persuasion case, if the information structure terminates at time $T$, then it reveals message $m_0$ with probability

$$q_0(T) = \begin{cases} cT & \text{if } \mu_0 > 1/2 \\ (1 - 2\mu_0)cT & \text{if } \mu_0 \leq 1/2 \end{cases}.$$  

Note that $T$ must be at most $\frac{\phi(\mu_0)}{c}$ since the pure persuasion case and the pure attention case coincide when $T = \frac{\phi(\mu_0)}{c}$. This means the sender must solve the optimization problem

$$\max_{T \leq \frac{\phi(\mu_0)}{c}} T + \lambda (1 - q_0(T)) = \begin{cases} \lambda + (1 - \lambda c)T, & \text{if } \mu_0 > 1/2 \\ \lambda + (1 - \lambda(1 - 2\mu_0)c)T, & \text{if } \mu_0 \leq 1/2 \end{cases}$$

Define $\lambda^*(\mu) = \frac{1}{c}$, if $\mu_0 > 1/2$ 

$\frac{1}{((1 - 2\mu_0)c)}$, if $\mu_0 \leq 1/2$. Then the sender’s optimal information structure is as follows:

1. If $\lambda < \lambda^*$, then the sender will send full information only once at time $\frac{\phi(\mu_0)}{c}$.

2. If $\lambda \geq \lambda^*$, then the sender will one-period persuade at time 1.

This result is intuitive. The more $\lambda$ is, the more the sender cares about a DM’s action, so the sender is more likely to choose the pure persuasion case.

A.5 Proof of Proposition 4. Before proving this proposition, we need the following lemma.

Lemma 5. Suppose that $f(\cdot, \cdot)$ is supermodular and $f(\cdot, 1)$ is concave. For every $p \in \mathcal{P}^{**}$, there is no $t^* > t \in \mathcal{T}$ such that $p_{1,t} > 0$, $\mu^{\rho}_t > \bar{\mu}$, and the DM is indifferent at $t^*$.

Proof of Lemma 5. Suppose a contradiction that there exists $p \in \mathcal{P}^{**}$ such that there exists $t^* > t \in \mathcal{T}$ such that $p_{1,t} > 0$, $\mu^{\rho}_t > \bar{\mu}$ and the DM is different at $t^*$. This implicitly implies the information structure ends after $t^*$. Suppose the feasible distribution $p$ is supported by an action-simple and obedient information structure $I$. With small $\delta$, we define information structure $I' \delta$ as follows:

1. $I'$ and $I$ coincide for every time $s < t - 1$: for any message $m$ and history $H_s$, $\mathcal{P}^I(m \mid H_s) = \mathcal{P}^{I'}(m \mid H_s)$.
2. At time $s = t - 1$ with history $(m^t_{\phi})^{t-1}$, we introduce a new message called $m_{\text{short}}$, which has the following information structure:

$$P^I(m_1 \mid (m^t_{\phi})^{t-1}) = P^I(m_1 \mid (m^t_{\phi})^{t-1}) - \delta$$

$$P^I(m_0 \mid (m^t_{\phi})^{t-1}) = P^I(m_0 \mid (m^t_{\phi})^{t-1})$$

$$P^I(m_{\phi} \mid (m^t_{\phi})^{t-1}) = P^I(m_{\phi} \mid (m^t_{\phi})^{t-1})$$

$$P^I(m_{\text{short}} \mid (m^t_{\phi})^{t-1}) = \delta.$$

3. At time $s = t^* - 1$ with history $(m^t_{\phi})^{t-1}$, we modify the information structure as follows:

$$P^{I'}(m_1 \mid (m^t_{\phi})^{t-1}) = P^I(m_1 \mid (m^t_{\phi})^{t-1}) + \delta$$

$$P^{I'}(m_0 \mid (m^t_{\phi})^{t-1}) = P^I(m_0 \mid (m^t_{\phi})^{t-1})$$

$$P^{I'}(m_{\phi} \mid (m^t_{\phi})^{t-1}) = P^I(m_{\phi} \mid (m^t_{\phi})^{t-1}) - \delta$$

4. $I'$ and $I$ coincide for every time $s \neq t^* - 1$ and history $(m^t_{\phi})$: $P^I(m \mid (m^t_{\phi})^t) = P^{I'}(m \mid (m^t_{\phi})^t)$.

5. For every time $s \geq t$ and history $((m^t_{\phi})^{t-1}, m_p, (m^s_{\phi})^{s-t})$, we use following information structure: $P^{I'}(m \mid ((m^t_{\phi})^{t-1}, m_p, (m^s_{\phi})^{s-t})) = P^I(m \mid (m^t_{\phi})^{s+1})$.

Intuitively speaking, we introduce $m_{\text{short}}$ at time $t - 1$ so that the information structure $I$ after $t^*$ and the information structure $I'$ after $t$ once the DM receives $m_{\text{short}}$ coincide. From this modified information structure, the posterior belief at $((m^t_{\phi})^{t-1}, m_p)$ under $I'$ is $\mu_{t+1}$, so $I'$ is well-defined. We can calculate that the information structure $I'$ induces a distribution $p'$ over $\Delta(\{0, \mu, 1\} \times \mathcal{T})$ such that, for some $\epsilon > 0$,

$$p'_{1,t} = p_{1,t} - \epsilon \sum_{m} \sum_{s=t+2}^\infty p_{m,s}.$$  

$$p'_{1,t+1} = p_{1,t+1} + \epsilon \sum_{m} \sum_{s=t+2}^\infty p_{m,s} + \epsilon p_{1,t+2}.$$  

$$p'_{0,t+1} = p_{0,t+1} + \epsilon p_{0,t+2}.$$  

$$p'_{m,s} = (1 - \epsilon)p_{m,s} + \epsilon p_{m,s+1} \quad \forall s > t + 1.$$

We now show that $p'$ satisfies obedience constraints. Consider that, for any $t' < t$,

$$\sum_{s=t'+1}^\infty \sum_{m \in \{0, \mu, 1\}} (p'_{m,s} - p_{m,s})(w'(m) - cs)$$

$$= -\sum_{s=t'+1}^\infty \sum_{m \in \{0, \mu, 1\}} (p'_{m,s} - p_{m,s})cs$$  

$$\quad (\sum_{s=t'+1}^\infty p'_{m,s} = \sum_{s=t'+1}^\infty p_{m,s} \text{ for every } m \in \{0, \mu, 1\})$$

$$= -\epsilon \sum_{m} \sum_{s=t+2}^\infty p_{m,s} - \epsilon (p_{1,t+2} + p_{0,t+2})(t + 1) - \epsilon \sum_{m} \sum_{s=t+2}^\infty (p_{m,s+1} - p_{m,s})s$$

$$= 0,$$

which implies that $\text{CONT}'_{t'} = \text{CONT}^p_{t'}$, and it is clear from the calculation above that $\text{STOP}'_{t'} = 23$. 

Electronic copy available at: https://ssrn.com/abstract=4595819
STOP$^\phi_t$ because the continuation belief at $t'$ does not change. Thus, the obedience constraint at time $t' < t$ holds. To show the obedience constraints hold for $t' \geq t$, we will show instead that $I'$ is obedient for every history $H_t$, where $t' \geq t$. The obedience constraints under $I$ and $I'$ at $(m_\delta)^t$ for every $t' > t$ coincide because they have the same future information structure. Similarly, the obedience constraint under $I'$ at $((m_\delta)^{t-1}, m_\rho, (m_\delta)^{t-1})$ and the obedience constraint under $I$ at $((m_\delta)^{t+1})$ coincide. Therefore, it is sufficient to consider the obedience constraint at $(m_\delta)^t$. The belief $\mu'_t$ after $(m_\delta)^t$ equals $\mu_t + \delta(1 - \mu_{t+1})$. Given the $(m_\delta)^t$, the continuation value under $I'$ is higher than that under $I$ by

$$\delta(u^*(1) - c(t + 1)) - \delta\text{CONT}^\phi_{t+1},$$

By the assumption we made at the beginning, $\text{CONT}^\phi_{t+1} = u^*(\mu_{t+1}) - c(t + 1)$ because the sender is indifferent at time $t^*$. Given the $(m_\delta)^t$, the continuation value under $I'$ is higher than that under $I$ by $\delta(u^*(1) - u^*(\mu_{t+1})) = \delta(1 - \mu_{t+1})u(1, 1)$ because $\mu_{t+1} \geq \bar{\mu}$. However, the continuation belief increases by $\delta(1 - \mu_{t+1})$, which means the utility of stopping at $t + 1$ increases by at most $\delta(1 - \mu_{t+1})|V_1|$.\(^7\) This implies the obedience constraint still holds at time between $t$ and $t^*$, as desired. Now we can compute the change in the sender’s utility:

$$\mathbb{E}'[f(a, \tau)] - \mathbb{E}[f(a, \tau)] = \sum_m \sum_{s=t+2}^\infty p_m s \left[ (f(1, t + 1) - f(1, t)) - (f(1_{m \geq \bar{\mu}}, s) - f(1_{m \geq \bar{\mu}}, s - 1)) \right] \delta$$

$$\geq \sum_m \sum_{s=t+2}^\infty p_m s \left[ (f(1, t + 1) - f(1, t)) - (f(1, s) - f(1, s - 1)) \right] \delta$$

> 0,

where the first and second inequalities follow by supermodularity of $f$ and concavity of $f(1, \cdot)$. This contradicts the optimality of $p$, as desired. \(\square\)

**Proof of Proposition 4.** First, we will show that there are at most two good signals in a sender’s optimal information structure. Suppose a contradiction that there are at least three good signals. Let the first three good signals are at time $t_1 < t_2 < t_3$. From Lemma 3, the DM must be indifferent at time $t_2$ and $\mu_{t_2} \geq \bar{\mu}$. However, this is a contradiction to the last lemma.

Because there are only two good signals, the information structure must terminate. We will show that the DM must be indifferent at every time before the information structure ends. We will show that 1) $\mu_t \geq \bar{\mu}$ for every $t > 0$, 2) the DM is indifferent at every continuation node, and 3) $p_{0,t} > 0$ for every $t > 0$ by backward induction.

**Basic step: $t = T - 1$.** Suppose a contradiction that the DM strictly prefers to pay attention at

\(^7\)Because $u^*$ is convex, we have

$$u^*(\mu_t + \delta(1 - \mu_{t+1})) - u^*(\mu_t) \leq \delta(1 - \mu_{t+1})(u^*)_\cdot(1) = \delta(1 - \mu_{t+1})u(1, 1),$$

where $(u^*)_\cdot(1)$ is the left derivative of $u^*$ at 1.
period $T - 1$. With sufficiently small $\delta$, we modify the information structure by

$$
\begin{align*}
\rho_{0,T-1}' &= \rho_{0,T-1} + \delta \\
\rho_{1,T}' &= \rho_{1,T} - (1 - \check{c})\delta \\
\rho_{0,T}' &= \rho_{0,T} - (1 + \check{c})\delta \\
\rho_{1,T+1}' &= (1 - \check{c})\delta \\
\rho_{0,T+1}' &= \check{c}\delta,
\end{align*}
$$

where $\check{c} = c/V_0$. The intuition of this modification is that the sender gives full information faster so that he can exploit the DM’s advantage by delaying action 1 to a later period with positive probability. It is easy to check that all probabilities of stopping nodes sum up to 1 as well as the martingale condition. It is also easy to check that the obedient constraint at time $T$ holds. The obedient constraint at time $T - 1$ still holds by a small perturbation because it slacks under the former information structure. Thus, it is sufficient to check that the continuation value after paying attention at time $T - 1$ is no worse than before. Note that the probability of receiving full information after time $T - 1$ (inclusive) does not change. Moreover, The change of the DM’s continuation cost after the time $T - 1$ is

$$\delta(T - 1)c - 2\delta T c + \delta(T + 1)c = 0,$$

which means the continuation value after time $T - 1$ is the same, implying the obedient constraint before time $T - 1$. This implies the modified information structure is simple and obedient. Using the linearity of $f(0, \cdot)$, the modified information structure increases the sender’s utility by $\delta(f(1, T + 1) - f(1, T))$. This contradicts the sender’s optimality, as desired.

Since the DM is indifferent at period $T - 1$, the belief and the continuation node at time $T - 1$ can be either $c$ or $1 - c$. If it becomes $c$, the DM’s indifference at period $T - 1$ implies nonzero probability of message 1 at period $T - 1$, which contradicts Lemma 3. Thus, the belief at time $T - 1$ must be $1 - c > \bar{\mu}$, which implies nonzero probability of message 0 at time $T - 1$.

Inductive step: Let $t_0$ be a number between $t$ and $T$ such that it is strictly suboptimal for the sender unless the DM must be indifferent at every time $t$ such that $t_0 < t < T$. We show earlier that the DM is indifferent at $T - 1$ and $\mu_{T-1}^p > \bar{\mu}$. The last lemma implies that $p_{1,t} = 0$ for every $t < T - 1$.

We will show that the previous statement is also true if $t = t_0$. Suppose a contradiction that the DM strictly prefers to pay attention at time $t_0$. We will modify the information structure such that

1. Introduce time $T + 1$: shift the probability of action 1 at time $T$ to time $T + 1$ while the obedience condition at time $T$ holds.

2. Maintain $CONT_t^p - STOP_t^p$ for every $t \in \{t_0 + 1, \ldots, T - 1\}$.

3. Maintain the continuation value $V_{t_0-1}^p$. 

Electronic copy available at: https://ssrn.com/abstract=4595819
To do this, let \((n_0, \ldots, n_{T+1})\) be a solution to the following system of linear equations:

\[
\begin{align*}
\sum_{t=0}^{T+1} n_t &= 0 \\
\sum_{t=0}^{T+1} n_t &= \frac{c}{v_0 - c} \\
\sum_{s=t}^{T+1} n_s &= \sum_{s=t}^{T+1} n_{s'} + 1 \quad \forall t \in \{t_0 + 2, \ldots, T\}, \\
\sum_{s=t}^{T+1} n_s &= 1.
\end{align*}
\]

(\ast)

It is easy to show that the above linear equations are linearly independent, so the existence of a solution is assured. With sufficiently small \(\delta > 0\), we modify the information structure by

1. \(p'_{0,t} = p_{0,t} + n_t \delta\) for every \(t \in \{t_0, \ldots, T+1\}\).
2. \(p'_{1,T} = p_{1,T} - \delta\) and \(p'_{1,T+1} = \delta\).

We will show that the modified probability distribution is valid and feasible.

1. \(p'_{m,t} \geq 0\) for every \(m\) and \(t\). It is sufficient to check only when \(m = 0\). Because \(n_{T+1} > 0\), \(p'_{0,T+1} > 0\). For every \(t \in \{t_0 + 2, \ldots, T\}\), the third equation of (\ast) implies that, if \(\sum_{s=t}^{T+1} n_{0,s'} > 0\) for every \(s > t\), then \(\sum_{s=t}^{T+1} n_{0,s'} > 0\). This together implies \(\sum_{s=t}^{T+1} n_{0,s'} > 0\) for every \(t \in \{t_0 + 2, \ldots, T\}\). Using the first and the fourth equations of (\ast), we obtain

\[
n_{0,t_0} = \sum_{s=t_0+2}^{T+1} n_{0,s}(s - t_0 - 1) + 1 > 0,
\]

which means \(p'_{0,t_0} > p_{0,t_0} \geq 0\). Because \(p_{0,t} > 0\) for every \(t \in \{t_0 + 1, \ldots, T\}\), we can choose small \(\delta\) so that \(p'_{0,t} > 0\) for every \(t \in \{t_0 + 1, \ldots, T\}\), as desired.

2. **Total probability and martingale conditions.** We can see that

\[
\begin{align*}
\sum_{t=t_0}^{T+1} p'_{0,t} &= \sum_{t=t_0}^{T+1} p_{0,t} + \sum_{t=t_0}^{T+1} n_t \delta = \sum_{t=t_0}^{T+1} p_{0,t} \\
\sum_{t=t_0}^{T+1} p'_{1,t} &= \sum_{t=t_0}^{T+1} p_{1,t},
\end{align*}
\]

which directly implies total probability and martingale conditions.

3. **Obedience condition at time \(T\).** We have

\[
\begin{align*}
STOP_T^{p'} &= p'_{1,T+1}u(1, 1) - c(p'_{0,T+1} + p'_{1,T+1})T, \\
\text{CONT}_T^{p'} &= p'_{0,T+1}u(0, 0) + p'_{1,T+1}u(1, 1) - c(p'_{0,T+1} + p'_{1,T+1})(T + 1)
\end{align*}
\]

and it is easily shown that \(STOP_T^{p'} = \text{CONT}_T^{p'}\) by the second equation of (\ast).
4. **Obedience condition at time** \( t \in \{t_0 + 1, \ldots, T - 1 \} \). We have

\[
STOP_t^{p'} - STOP_t^p = (\Delta p_{1,T} + \Delta p_{1,T+1})V_1 - c t (\sum_{s=t}^{T+1} \Delta p_{0,s} + \Delta p_{1,T} + \Delta p_{1,T+1})
\]

\[
CONT_t^{p'} - CONT_t^p = - \sum_{s=t}^{T+1} \Delta p_{0,s} c s - \Delta p_{1,T} c T - \Delta p_{1,T+1} c (T+1),
\]

where \( \Delta p_{m,t} := p'_{m,t} - p_{m,t} \). From the third equation of (*), it is easy to show that \( STOP_t^{p'} - STOP_t^p = CONT_t^{p'} - CONT_t^p \). Thus, \( CONT_t^{p'} - STOP_t^p = CONT_t^p - STOP_t^p \geq 0 \), as desired.

5. **Obedience condition at time** \( t_0 \). Since the obedience constraint at time \( t_0 \) under \( p \) slacks, there exists a sufficiently small \( \delta \) which still maintains the obedience constraint at time \( t_0 \), as desired.

6. **Obedience condition at time** \( t < t_0 \). Since the continuation belief at time \( t < t_0 \) does not change, it is sufficient to show that the continuation value at time \( t_0 \) weakly increases. We can see that

\[
CONT_t^{p'} - CONT_t^p = \sum_{s=t}^{T+1} \Delta p_{0,s} c s + \Delta p_{1,T} c T + \Delta p_{1,T+1} c (T+1) = 0
\]

from the fourth equation of (*), as desired.

These together imply \( p' \) is a feasible distribution. Under this modified information structure, the sender’s utility increases by

\[
\mathbb{E}^{p'}[f(a, \tau)] - \mathbb{E}^p[f(a, \tau)] = \sum_{s=t_0}^{T+1} \Delta p_{0,s} f(0, s) + \delta(f(1, T+1) - f(1, T))
\]

\[
= \sum_{s=t_0}^{T+1} \Delta p_{0,s} (f(0, s) - f(0, 0)) + \delta(f(1, T+1) - f(1, T))
\]

\[
= \sum_{s=t_0}^{T+1} s \Delta p_{0,s} (f(0, T+1) - f(0, T)) + \delta(f(1, T+1) - f(1, T))
\]

\[
= -\delta(f(0, T+1) - f(0, T)) + \delta(f(1, T+1) - f(1, T)) > 0,
\]

which contradicts the optimality of \( p \). Therefore, the DM must be indifferent at time \( t_0 \). Suppose that \( \mu_{t_0} \leq \bar{\mu} \), which implies a nonzero probability of message 1 at period \( t_0 \). This contradicts what we prove earlier that \( p_{1,t} = 0 \) for every \( t < T - 1 \). Therefore, \( \mu_{t_0} > \bar{\mu} \), which implies a nonzero probability of message 0 at time \( t_0 \), as desired. Lemma 5 directly implies that, if there are two good signals, then they have to be at time \( T - 1 \) and \( T \).

Lastly, we will show that the obedient constraint at time 0 must bind if \( f(0, \cdot) \) is weakly increasing. Suppose towards a contradiction that there is \( p \in \mathcal{P}^* \) such that the obedient constraint at time 0 does not bind. Suppose \( p \) is supported by an extreme-simple and obedient information structure \( I \). With small \( \epsilon \), we define an information structure \( I' \) as follows:
1. At time 0, we introduce a new message called $m_{\text{delay}}$, which has the following information structure:

$$
\mathbb{P}^{I'}(m_i \mid 0) = (1 - \epsilon)\mathbb{P}^{I}(m_i \mid 0)
$$

$$
\mathbb{P}^{I'}(m_{\text{delay}} \mid 0) = \epsilon \sum_{i \in \{0,1,0\}} \mathbb{P}^{I}(m_i \mid 0),
$$

for every $i \in \{0, 1, 0\}$.

2. At time $t > 0$ with any history $H_t$ that does not contain $m_{\text{delay}}$, $I$ and $I'$ coincide: $\mathbb{P}^{I'}(m \mid H_t) = \mathbb{P}^{I}(m \mid H_t)$ for every message $m$.

3. At time $t > 0$ with any history $(m_{\text{delay}}, H_{t-1})$, $\mathbb{P}^{I'}(m \mid (m_{\text{delay}}, H_{t-1})) = \mathbb{P}^{I}(m \mid H_{t-1})$ for every message $m$.

Intuitively speaking, this information structure coincides with $I$ with probability $1 - \epsilon$ and one-period delayed $I$ with probability $\epsilon$. This information structure is well-defined because the posterior belief after receiving $m_{\text{delay}}$ is still $\mu_0$. The obedient constraints of $I'$ at time $t > 0$ clearly hold because of the obedient constraints of $I$. The obedient constraint at time 0 of $I'$ also holds with small $\epsilon$. Therefore, $I'$ is obedient. However, the sender’s utility under $I'$ is

$$(1 - \epsilon) \sum_{m=0,1} \sum_{t \in T} p_{m,t} f(m, t) + \epsilon \sum_{m=0,1} \sum_{t \in T} p_{m,t+1} f(m, t) > \sum_{m=0,1} \sum_{t \in T} p_{m,t} f(m, t)$$

because $f(0, \cdot)$ and $f(1, \cdot)$ are increasing, which contradicts the optimality of $p$, as desired. ∎
Appendix B: Proof of extremal reduction principle

Suppose the DM’s action space $A$ is finite. Fix a DM’s utility function $v : A \times \Theta \rightarrow \mathbb{R}$. Define the optimal DM’s utility function $v^* : \Delta(\Theta) \rightarrow \mathbb{R}$ and the DM’s best response correspondence $a^* : \Delta(\Theta) \rightrightarrows A$ as follows:

$$v^*(\mu) = \max_{a \in A} \mathbb{E}_{\theta \sim \mu} v(a, \theta), \quad a^*(\mu) = \arg \max_{a \in A} \mathbb{E}_{\theta \sim \mu} v(a, \theta).$$

For each $A' \subset A$, define the set of beliefs under which every action in $A'$ is one of the DM’s best responses:

$$\Theta_{A'} := \{ \mu \in \Delta(\Theta) : A' \subset a^*(\mu) \}.$$  

Since $A$ is finite, $\Theta_{A'}$ is a polytope. If $A' = \{a\}$, we abuse a notation $\Theta_a$ with $\Theta_{A'}$. We call a map $\gamma : \Delta(\Theta) \rightarrow \Delta(A)$ a proper tie-breaking rule under $v$ if $\mu \in \Theta_a$ for every belief $\mu \in \Delta(\Theta)$ and action $a \in A$ such that $\gamma(\mu)(a) > 0$. We define a distribution $d_\mu$ over $A$ is induced by a pair of a signal structure and a proper tie-breaking rule $(X, \gamma)$, where $X \in \Delta(\Delta(\Theta))$ if $d_\mu(a) = \mathbb{E}_{\mu \sim X}[\gamma(\mu)(a)]$ for every $a \in A$.

We call $\mu$ a sufficient stopping belief under a set of actions $A'$ if and only if $\mu$ is an extreme point of $\Theta_{A'}$. We denote $\Delta^*_{A'}(\Theta)$ as the set of sufficient stopping beliefs a set of actions $A'$ and $\Delta^*(\Theta)$ as the set of sufficient stopping beliefs.

Every belief $\mu \in \Delta(\Theta)$ can be written as a convex combination of sufficient stopping beliefs under the set of actions $a^*(\mu)$ as follows

$$\mu = \sum_{\mu^* \in \Delta^*_{a^*(\mu)}(\Theta)} \alpha_{\mu}(\mu^*) \mu^*$$

for a unique mapping $\alpha_{\mu} : \Delta^*_{a^*(\mu)}(\Theta) \rightarrow \mathbb{R}^+$ such that $\sum_{\mu^* \in \Delta^*_{a^*(\mu)}(\Theta)} \alpha_{\mu}(\mu^*) = 1$.

For any signal structure $X \in \Delta(\Delta(\Theta))$, we define a sufficient signal structure of $X$ as $X^* \in \Delta(\Delta^*(\Theta))$ such that, for every $\mu^* \in \Delta^*(\Theta)$,

$$X^*(\mu^*) = \mathbb{E}_{\mu \sim X}[\alpha_{\mu}(\mu^*)],$$

where $\alpha_{\mu}(\mu^*) = 0$ if $\mu^* \notin \Delta^*_{a^*(\mu)}(\Theta)$. This is a well-defined probability distribution because

$$\sum_{\mu^* \in \Delta^*(\Theta)} X^*(\mu^*) = \sum_{\mu^* \in \Delta^*(\Theta)} \mathbb{E}_{\mu \sim X}[\alpha_{\mu}(\mu^*)] = \mathbb{E}_{\mu \sim X} \left[ \sum_{\mu^* \in \Delta^*(\Theta)} \alpha_{\mu}(\mu^*) \right] = 1.$$

**Lemma 6.** Suppose $X^*$ is a sufficient signal structure of $X \in \Delta(\Delta(\Theta))$ and $\gamma$ is a proper tie-breaking rule under $v$. Then

1. (Mean-preserving)

$$\sum_{\mu^* \in \Delta^*(\Theta)} X^*(\mu^*) \mu^* = \mathbb{E}_{\mu \sim X}[\mu]$$

2. The DM’s expected utilities under $X$ and $X^*$ are exactly the same.

3. There exists a proper tie-breaking rule $\gamma^*$ under $v$ such that $(X, \gamma)$ and $(X^*, \gamma^*)$ induce
the same distribution of actions.

**Proof.** Consider that

$$
\sum_{\mu^* \in \Delta^*(\Theta)} X^*(\mu^*)\mu^* = \sum_{\mu^* \in \Delta^*(\Theta)} \mathbb{E}_{\mu^*}[\alpha_\mu(\mu^*)]\mu^* = \mathbb{E}_{\mu^*}[\sum_{\mu^* \in \Delta^*(\Theta)} \alpha_\mu(\mu^*)\mu^*] = \mathbb{E}_{\mu^*}[\mu],
$$

which implies the first statement. For the second statement, consider that

$$
\sum_{\mu^* \in \Delta^*(\Theta)} X^*(\mu^*)\varphi^*(\mu^*) = \sum_{\mu^* \in \Delta^*(\Theta)} \mathbb{E}_{\mu^*}[\alpha_\mu(\mu^*)]\varphi^*(\mu^*)
= \mathbb{E}_{\mu^*}[\sum_{\mu^* \in \Delta^*(\Theta)} \alpha_\mu(\mu^*)\varphi^*(\mu^*)]
= \mathbb{E}_{\mu^*}[\sum_{\mu^* \in \Delta^*(\Theta)} \alpha_\mu(\mu^*)] \mathbb{E}_{\mu^*}[\varphi^*(\alpha^*, \mu^*)]
= \mathbb{E}_{\mu^*}[\varphi^*(\mu^*)],
$$

as desired. For the third statement, we choose a tie-breaking rule $\gamma^*: \Delta(\Theta) \rightarrow \Delta(A)$ as follows:

$$
\gamma^*[\mu^*](a) = \frac{\mathbb{E}_{\mu^*}[\alpha_\mu(\mu^*)\varphi(\mu^*)]}{X^*(\mu^*)},
$$

for every $\mu^* \in \Delta^*(\Theta)$ and $a \in A$ such that $X^*(\mu^*) > 0$. If either $\mu^* \notin \Delta^*(\Theta)$ or $\mathbb{E}_{\mu^*}[\alpha_\mu(\mu^*)] = 0$, assume $\gamma^*[\mu] = \varphi(\mu)$.

This is well-defined because

$$
\sum_{a \in A} \gamma^*[\mu^*](a) = \frac{\mathbb{E}_{\mu^*}[\alpha_\mu(\mu^*) \sum_{a \in A} \varphi(\mu^*)]}{X^*(\mu^*)} = \frac{\mathbb{E}_{\mu^*}[\alpha_\mu(\mu^*)]}{\mathbb{E}_{\mu^*}[\alpha_\mu(\mu^*)]} = 1.
$$

We show that $\gamma^*$ is a proper tie-breaking rule. If $\gamma^*[\mu^*](a) > 0$, then there exists $\mu \in \Delta^*(\Theta)$ such that $\alpha_\mu(\mu^*) > 0$ and $\varphi(\mu^*) > 0$. This implies $\mu^* \in \Delta^*_a(\Theta)$ and $a \in \alpha^*(\mu^*)$, which means $a \in a^*(\mu) \subseteq a^*(\mu^*)$, as desired. For every action $a \in A$, we have

$$
\sum_{\mu^* \in \Delta^*(\Theta)} X^*(\mu^*)\gamma^*[\mu^*](a) = \sum_{\mu^* \in \Delta^*(\Theta)} X^*(\mu^*) \cdot \frac{\mathbb{E}_{\mu^*}[\alpha_\mu(\mu^*)\gamma(\mu^*)]}{X^*(\mu^*)}
= \mathbb{E}_{\mu^*}[\sum_{\mu^* \in \Delta^*(\Theta)} \alpha_\mu(\mu^*)\gamma(\mu)^*(a)]
= \mathbb{E}_{\mu^*}[\gamma(\mu)(a)],
$$

as desired.

Intuitively speaking, a sufficient signal structure has a support of the set of sufficient stopping beliefs and induces the same distribution of best responses with some tie-breaking rule $\gamma^*$. 

30
Consider the setting of dynamic information structures. Fix a DM’s utility function \( v : A \times \Theta \times \mathcal{T} \to \mathbb{R} \). We define \( v_t : A \times \Theta \to \mathbb{R} \) as the DM’s utility function at time \( t \): \( v_t(\cdot, \cdot) = v(\cdot, \cdot, t) \). We call a sequence of maps \((\gamma_t)\), where \( \gamma_t : \Delta(\Theta) \to \Delta(A) \) is a proper tie-breaking rule under \( v_t \), a sequence of proper tie-breaking rules under \( v \). Note that this implicitly requires tie-breaking rules depending only on time \( t \), not the history \( H_t \). Since the probability distribution of DM’s action depends on tie-breaking rules, we sometimes write the probability operator and the expectation operator as \( \mathbb{P}^I(\cdot) \) and \( \mathbb{E}^I[\cdot] \), where \( I \) and \( \gamma = (\gamma_t)_{t \in \mathcal{T}} \) are the corresponding information structure and the sequence of tie-breaking rules, respectively.

We say a distribution \( d \) over \( A \times \Theta \) is induced by a pair of a dynamic information structure and a sequence of proper tie-breaking rules \((I, (\gamma_t))\) if

\[
d_A(a, t) = \mathbb{E}^I[\gamma_t(\mu_t)(a) \cdot 1\{\tau(I) = t\}],
\]

where \( \mu_t \) is a (random) posterior belief at time \( t \), for every \( a \in A \) and \( t \in \mathcal{T} \).

**Proposition 5** (Extremal reduction belief principle). For every dynamic information structure \( I \) with a tie-breaking rule \((\gamma_t)\), there exists an extremal and obedient information structure \( I' \) and a sequence of proper tie-breaking rules \((\gamma'_t)\), under the DM’s utility function \( v \) such that \((I, (\gamma_t))\) and \((I', (\gamma'_t))\) yield the same joint distribution of the stopping time and action.

Proposition 5 implies Proposition 1. We mimic the proof of Proposition 1 from Koh and Sanganmoo (2022) as follows:

Consider a dynamic information structure \( I \). For every optimal stopping history \((m_1, \ldots, m_t) \in \mathcal{H}^O(I) \), we relabel \( m_{t} \) by \( m_\mu \), where \( \mu \) is the DM’s posterior belief after history \((m_1, \ldots, m_t) \).

**Step 1: Collapse non-optimal stopping histories \( \mathcal{H}^E(I) \backslash \mathcal{H}^O(I) \).**

We construct a simple information structure

\[
I' = \left\{ p^I_\mu(\cdot \mid \theta, H_t) \right\}_{t \in \mathcal{T}, \theta \in \Theta, H_t \in \mathcal{H}_t}
\]

with the message space \( M' = \{m_\theta\} \cup \{m_\mu\}_{\mu \in \Delta(\Theta)} \) as follows. Define a new static information structure at history \((m_\theta)^t \) (the history comprising the message \( m_\theta \) for \( t \) consecutive periods) such that for every state \( \theta \in \Theta \) and belief \( \mu \in \Delta(\Theta) \),

\[
\begin{align*}
(i) \quad & p^I_\mu(m_\mu \mid \theta, (m_\theta)^t) = \frac{\sum_{H_t \in \mathcal{H}^O_t \mathcal{H}^O_t} \mathbb{P}^I(H_t \mid \theta) p_\mu^I(m_\mu \mid \theta, H_t)}{\sum_{H_t \in \mathcal{H}^O_t \mathcal{H}^O_t} \mathbb{P}^I(H_t \mid \theta)} \\
(ii) \quad & p^I_\mu(m_\theta \mid \theta, (m_\theta)^t) = \frac{\sum_{H_{t+1} \in \mathcal{H}^O_{t+1} \mathcal{H}^O_t} \mathbb{P}^I(H_{t+1} \mid \theta)}{\sum_{H_t \in \mathcal{H}^O_t \mathcal{H}^O_t} \mathbb{P}^I(H_t \mid \theta)}
\end{align*}
\]

and one can check that this dynamic information structure is well-defined i.e., \( I' \in I^* \). Part (i) takes the probability under \( I \) that the DM receives message \( m_\mu \) at time \( t+1 \), conditional on not stopping before \( t \), and equates that to the probability under \( I' \) that she receives the message \( m_\mu \) at time \( t+1 \), conditional on receiving only the message \( m_\theta \) before \( t \). Part (ii) is the complement of part (i) and takes the probability under \( I \) that the DM does not stop at time

---

\(^8\mathcal{H}^E(I) \) and \( \mathcal{H}^O(I) \) are defined in the proof of Proposition 1 from Koh and Sanganmoo (2022)
for each

so by Bayes’ rule,

\[
\mathbb{P}^I((m_\theta)^t | \theta) = \prod_{s=0}^{t-1} p^I_s(m_\theta | \theta, (m_\theta)^s)
\]

\[
= \sum_{H_t \in \mathcal{H}_t} \mathbb{P}^I(H_t | \theta)
\]

\[
= \mathbb{P}^I(\tau(I) > t | \theta),
\]

so by Bayes’ rule,

\[
\mathbb{P}^I(\theta | (m_\theta)^t) = \frac{\mu_0(\theta)\mathbb{P}^I((m_\theta)^t | \theta)}{\sum_{\theta' \in \Theta} \mu_0(\theta')\mathbb{P}^I((m_\theta)^t | \theta)}
\]

\[
= \frac{\mu_0(\theta)\mathbb{P}^I(\tau(I) > t | \theta)}{\sum_{\theta' \in \Theta} \mu_0(\theta')\mathbb{P}^I(\tau(I) > t | \theta')}
\]

\[
= \mathbb{P}^I(\theta | \tau(I) > t),
\]

for each \(\theta \in \Theta\) as desired. Moreover,

\[
\mathbb{P}^I((m_\theta)^t, m_\mu | \theta) = \prod_{s=0}^{t-1} p^I_s(m_\theta | \theta, (m_\theta)^s)p^I_s(m_\mu | \theta, (m_\theta)^s)
\]

\[
= \sum_{H_t \in \mathcal{H}_t} \mathbb{P}^I((H_t, m_\mu) | \theta),
\]

so by Bayes’ rule

\[
\mathbb{P}^I(\theta | (m_\theta)^t, m_\mu) = \frac{\mu_0(\theta)\mathbb{P}^I((m_\theta)^t, m_\mu | \theta)}{\sum_{\theta' \in \Theta} \mu_0(\theta')\mathbb{P}^I((m_\theta)^t, m_\mu | \theta)}
\]

\[
= \frac{\sum_{H_t \in \mathcal{H}_t} \mu_0(\theta)\mathbb{P}^I((H_t, m_\mu) | \theta)}{\sum_{\theta' \in \Theta} \sum_{H_t \in \mathcal{H}_t} \mu_0(\theta')\mathbb{P}^I((H_t, m_\mu) | \theta')}
\]

\[
= \mu(\theta)
\]

because, for every history \(H_t\),

\[
\mu(\theta) = \mathbb{P}^I(\theta | (H_t, m_\mu)) = \frac{\mu_0(\theta)\mathbb{P}^I((H_t, m_\mu) | \theta)}{\sum_{\theta' \in \Theta} \mu_0(\theta')\mathbb{P}^I((H_t, m_\mu) | \theta')}.
\]
The next lemma establishes that if, under \( I' \), the DM has not yet received message \( m_\mu \) until time \( t \), her expected utility from not stopping until seeing message \( m_\mu \) for some \( \mu \in \Delta(\Theta) \) is the same if she followed the same strategy under \( I \). This follows because (i) by Lemma 7, the DM’s belief under \( I \) and \( I' \) at time \( t \) conditional on not stopping is identical; and (ii) the probabilities of receiving message \( m_\mu \) in future periods conditional on not yet receiving message \( m_\mu \) until time \( t \) is identical by the construction of \( I' \).

**Lemma 8.** If the DM never stops until seeing a message \( m_\mu \) under \( I' \), then his expected utility given history \( (m_\theta)^I \) is \( U''(m_\theta)^I = \mathbb{E}^I[u^*((\mu_0)H_{\tau(I)}, \tau)|\tau(I) > t] \), where \( u^*(\mu, t) := \max_{a \in A} \mathbb{E}_{\theta \sim \mu^*}(a, \theta, t) \) for every \( \mu \in \Delta(\Theta) \) and \( t \in \mathcal{T} \).

**Proof.** For any \( \theta \in \Theta, t \in \mathcal{T}, \) and \( s > t \), the probability of history \((m_\theta)^{s-1}, m_\mu)\) given belief \( \mu \), history \((m_\theta)^I\), and state \( \theta \) is

\[
P''((m_\theta)^{s-1}, m_\mu)|\theta, (m_\theta)^I) = p''_t(m_\mu | \theta, (m_\theta)^{s-1}) \times \prod_{s'=t}^{s-2} p''_{s'}(m_\theta | \theta, (m_\theta)^{s'})
\]

\[
= \frac{\sum_{H_s \in \mathcal{H}^I_t(H)} P^I(\theta, H_s)1(m_s = m_\mu)}{\sum_{H_s \in \mathcal{H}^I_t(H)} P^I(H_s | \theta)}
\]

\[
= P^I(\tau(I) = s, m_s = m_\mu | \theta, \tau(I) > t).
\]

This implies

\[
P''((m_\theta)^{s-1}, m_\mu)|(m_\theta)^I) = P^I(\tau(I) = s, m_s = m_\mu | \tau(I) > t).
\]

The DM’s expected utility given history \((m_\theta)^I\) is conditioned on her belief \( \mu_0 |(m_\theta)^I \in \Delta(\Theta)\):

\[
U''((m_\theta)^I) = \sum_{s=t+1}^{+\infty} \sum_{\mu \in \Delta(\Theta)} P''((m_\theta)^{s-1}, m_\mu) | (m_\theta)^I) u^*(\mu, s)
\]

\[
= \sum_{s=t+1}^{+\infty} \sum_{\mu \in \Delta(\Theta)} P^I(\tau(I) = s, m_s = m_\mu | \tau(I) > t)u^*(\mu, s)
\]

\[
= \mathbb{E}^I[u^*(\mu_0)H_{\tau(I)}, \tau(I) | \tau(I) > t],
\]

as desired. The first equality used Lemma 7, which implies \( \mu \) is a posterior belief of the history \((m_\theta)^{s-1}, m_\mu)\) under \( I' \). The second equality used the result developed above that the conditional distribution of the stopping time \( \tau(I) \) under \( I \) is identical to that of the time at which she receives message \( m_\mu \) under \( I' \).

The next Lemma states that this modification from \( I \) to \( I' \) remains obedient.

**Lemma 9.** Under \( I' \), for any \( t \in \mathcal{T} \), the DM continues paying attention at history \((m_\theta)^I\).

**Proof.** It will suffice to show that the DM does weakly better by continuing until seeing message \( m_\mu \) for some \( \mu \in \Delta(\Theta) \) since that is a stopping time the DM can choose. To this end, note
that the DM’s expected utility if she stops at history \((m_0)^I\) under \(I'\) is

\[
\max_{a \in A} \mathbb{E}^{I'}_{a} [v(a, \theta, t) | (m_0)^I] = \max_{a \in A} \mathbb{E}^{I} [v(a, \theta, t) | \tau(I) > t] \tag{Lemma 7}
\]

\[
= \max_{a \in A} \mathbb{E}^{I} [\mathbb{E}^{I} [v(a, \theta, t) | H_t] | \tau(I) > t]
\]

\[
(H_t \in \mathcal{H}_I^I(I) \setminus \mathcal{H}_I^O(I) \text{ is a R.V.})
\]

\[
\leq \mathbb{E}^{I} \left[ \max_{a \in A} \mathbb{E}^{I} [v(a, \theta, t) | H_t] | \tau(I) > t \right] \tag{Jensen}
\]

\[
\leq \mathbb{E}^{I} [U^I(H_t) | \tau(I) > t] \tag{Def. of \(U^I(\cdot)\))
\]

\[
= \mathbb{E}^{I} \left[ \mathbb{E}^{I} [v'(\mu_0|H_t(I), \tau(I)) | H_t] | \tau(I) > t \right] \tag{Iterated expectation}
\]

\[
= U^{I'}((m_0)^I) \tag{Lemma 8}
\]

which implies that the DM does weakly better by continuing to pay attention. \(\square\)

From Lemma 9, under \(I'\) the DM does not stop unless she sees message \(m_\mu\) for some \(\mu \in \Delta(\Theta)\) i.e., \(\mathbb{P}^{I'}((\tau(I') = s, m_s = m_\mu) = \mathbb{P}^{I'}((m_0)^{t-1}, m_\mu)).\) From the first part of the proof of Lemma 8, the probability that the DM stops paying attention at time \(t\) with a posterior belief \(\mu \in \Delta(\Theta)\) is

\[
\mathbb{P}^{I'}((m_0)^{t-1}, m_\mu) = \mathbb{P}^{I}(\tau(I) = t, m_t = m_\mu).
\]

This implies, for every action \(a \in A\) and time \(t \in T',\)

\[
\mathbb{P}^{I,y}(\tau(I) = t, a_{\tau(I)} = a) = \sum_{\mu \in \Delta(\Theta)} \mathbb{P}^{I}(\tau(I) = t, m_t = m_\mu) y_I[\mu](a)
\]

\[
= \sum_{\mu \in \Delta(\Theta)} \mathbb{P}^{I'}((m_0)^{t-1}, m_\mu) y_I[\mu](a)
\]

\[
= \mathbb{P}^{I,y'}(\tau(I) = t, a_{\tau(I)} = a),
\]

which implies \(I\) and \(I'\) yield the same joint distribution of action and stopping time under the same sequence of tie-breaking rules \((y_I)_{t \in T'}\).

**Step 2: Modify optimal stopping histories on the information structure \(I'\)**

For every \(t \in T\), define \(X_t \in \Delta(\Delta(\Theta))\) as a static information structure of \(I'\) conditional on stopping at time \(t\), i.e. for every \(\mu \in \Delta(\Theta),\)

\[
X_t(\mu) = \mathbb{P}^{I'}(m_t = m_\mu | \tau(I') = t).
\]

Let \(X_* \in \Delta(\Delta^*(\Theta))\) be a sufficient signal structure of \(X_t\). From Lemma 6, there exists a proper tie-breaking rule \(y_*^I\) such that \((X_t, y_I)\) and \((X_*^t, y_*^I)\) induce the same distribution of actions under \(v_t\). Construct a new information structure \(I''\) from \(I'\) as follows:

(i) For all times \(t\) and extremal beliefs \(\mu \in \Delta^*(\Theta),\) set

\[
p^{I''}(m_\mu | \theta, (m_0)^t) = \frac{X_*^{t+1}(\mu) \mu(\theta)}{\sum_{\mu' \in \Delta^*(\Theta)} X_*^{t+1}(\mu') \mu'(\theta)} \cdot \mathbb{P}^{I'}(m_{t+1} \neq m_0 | \theta, (m_0)^t).
\]

34
(ii) For all \( m \in M \) and \( \theta \in \Theta \),
\[
p_t^{I''}(m_\theta \mid \theta, (m_\theta)^i) = p_t^{I''}(m_\theta \mid \theta, (m_\theta)^i)
\]

Part (i) says \( I'' \) redistributes stopping messages with arbitrary posteriors belief \( \mu \) from \( I' \) into those with sufficient beliefs. One can check the resultant information structure \( I'' \) is well-defined.

**Lemma 10.** \( \mathbb{P}^{I''}(\theta \mid (m_\theta)^i) = \mathbb{P}^{I'}(\theta \mid (m_\theta)^i) \) for all \( \theta \in \Theta \). Moreover, for every \( \mu \in \Delta^*(\Theta) \), if \( \mathbb{P}^{I''}(\mu \mid (m_\theta)^i) > 0 \), then \( \mathbb{P}^{I''}(\theta \mid ((m_\theta)^i, \mu)) = \mu(\theta) \) for every \( \theta \in \Theta \).

**Proof of Lemma 10.** Observe
\[
\mathbb{P}^{I''}((m_\theta)^i \mid \theta) = \prod_{s=0}^{t-1} p_s^{I''}(m_\theta \mid \theta, (m_\theta)^i) = \prod_{s=0}^{t-1} p_s^{I'}(m_\theta \mid \theta, (m_\theta)^i) = \mathbb{P}^{I'}((m_\theta)^i \mid \theta).
\]
so by Bayes’ rule,
\[
\mathbb{P}^{I''}(\theta \mid (m_\theta)^i) = \frac{\sum_{\theta' \in \Theta} \mu(\theta') \mathbb{P}(\theta' \mid (m_\theta)^i)}{\sum_{\theta' \in \Theta} \mu(\theta') \mathbb{P}(\theta' \mid (m_\theta)^i)} = \mathbb{P}^{I'}(\theta \mid (m_\theta)^i),
\]
for each \( \theta \in \Theta \), which implies the first statement of the lemma. From Lemma 6,
\[
\sum_{\mu' \in \Delta^+(\Theta)} X_{t+1}^{\ast}(\mu') \mu'(\theta) = \sum_{\mu' \in \Delta^+(\Theta)} X_{t+1}(\mu') \mu'(\theta)
\]
\[
= \sum_{\mu' \in \Delta^+(\Theta)} \mathbb{P}^{I'}(m_{t+1} = m_{\mu'} \mid \tau(I') = t \mid \theta) \mathbb{P}^{I'}(m_{t+1} = m_{\mu'}, \tau(I') = t + 1)
\]
\[
= \mathbb{P}^{I'}(\theta \mid \tau(I') = t + 1).
\]
This implies
\[
p_t^{I''}(\mu \mid \theta, (m_\theta)^i) = \frac{X_{t+1}^{\ast}(\mu) \mu(\theta)}{\mathbb{P}^{I'}(\theta \mid \tau(I') = t + 1)} \cdot \mathbb{P}^{I'}(\theta \mid \tau(I') = t + 1, (m_\theta)^i).
\]
This means, for every \( \theta \in \Theta \) and \( \mu \in \Delta(\Theta) \),
\[
\mathbb{P}^{I''}((m_\theta)^i, \mu, \theta) = \mu(\theta) \prod_{s=0}^{t} p_s^{I''}(m_\theta \mid \theta, (m_\theta)^i) \cdot p_t^{I''}(\mu \mid \theta, (m_\theta)^i)
\]
\[
= \mu(\theta) \prod_{s=0}^{t} p_s^{I'}(m_\theta \mid \theta, (m_\theta)^i) \cdot \frac{X_{t+1}^{\ast}(\mu) \mu(\theta)}{\mathbb{P}^{I'}(\theta \mid \tau(I') = t + 1)} \cdot \mathbb{P}^{I'}(\tau(I') = t + 1 \mid \theta, (m_\theta)^i)
\]
\[
= \frac{X_{t+1}^{\ast}(\mu) \mu(\theta)}{\mathbb{P}^{I'}(\theta \mid \tau(I') = t + 1)} \cdot \mathbb{P}^{I'}(\tau(I') = t + 1, \theta)
\]
\[
= X_{t+1}^{\ast}(\mu) \mathbb{P}^{I'}(\tau(I') = t + 1) \cdot \mu(\theta).
\]
Thus,
\[ \mathbb{E}^{I''}((m_0)^t, m_\mu)) = \sum_{\theta \in \Theta} X^*_t+1(\mu)\mathbb{E}^{I'}(\tau(I') = t+1) \cdot \mu(\theta) = X^*_t+1(\mu)\mathbb{E}^{I'}(\tau(I') = t+1). \]

By Bayes’ rule,
\[ \mathbb{E}^{I''}(\theta | ((m_0)^t, m_\mu)) = \frac{X^*_t+1(\mu)\mathbb{E}^{I'}(\tau(I') = t+1) \cdot \mu(\theta)}{X^*_t+1(\mu)\mathbb{E}^{I'}(\tau(I') = t+1)} = \mu(\theta), \]
as desired.

\textbf{Lemma 11.} For all times \( t \) and extremal beliefs \( \mu \in \Delta^*(\Theta) \), we have
\[ p^I''_t (m_\mu | (m_0)^t) = X^*_t+1(\mu) \cdot \mathbb{E}^{I'}(m_{t+1} \neq m_0 | (m_0)^t). \]

\textbf{Proof of Lemma 11.} We show earlier that
\[ p^I''_t (m_\mu | \theta, (m_0)^t) = \frac{X^*_t+1(\mu)\mu(\theta)}{\mathbb{E}^{I'}(\tau(I') = t+1)} \cdot \mathbb{E}^{I'}(\tau(I') = t+1 | \theta, (m_0)^t), \]
which implies
\[ p^I''_t (\theta, m_\mu | (m_0)^t) = X^*_t+1(\mu)\mu(\theta) \cdot \mathbb{E}^{I'}(\tau(I') = t+1 | (m_0)^t). \]

Thus,
\[ p^I''_t (m_\mu | (m_0)^t) = \sum_{\mu \in \Delta^*(\Theta)} X^*_t+1(\mu)\mu(\theta) \cdot \mathbb{E}^{I'}(\tau(I') = t+1 | (m_0)^t) = X^*_t+1(\mu) \cdot \mathbb{E}^{I'}(\tau(I') = t+1 | (m_0)^t), \]
as desired.

Further define \( U^I(H_t) \) as the expected utility of DM at history \( H_t \) under the information structure \( I \in \mathcal{I} \) if she does not stop until she sees message \( m_\mu \) for some \( \mu \in \Delta(\Theta) \).

\textbf{Lemma 12.} Under \( I'' \), for any \( t \in \mathcal{T} \), the DM continues paying attention at history \( (m_0)^t \).

\textbf{Proof of Lemma 12.} Suppose \( \bar{\tau}(I'') \) be a stopping time if the DM continues paying attention until seeing message \( m_\mu \) for some \( \mu \). The condition (ii) implies that, for every \( t \in \mathcal{T} \),
\[ \mathbb{F}(\bar{\tau}(I'') > t) = \prod_{s=0}^{t-1} p^I_{s+1} (m_0 | (m_0)^t) = \prod_{s=0}^{t-1} p^I''_s (m_0 | (m_0)^t) = \mathbb{P}(\tau(I') > t). \]

This implies \( \tau(I') \) and \( \bar{\tau}(I'') \) equal in distribution. From Lemma 11, for every extremal belief \( \mu \in \Delta^*(\Theta) \),
\[ p^I''_t (m_\mu | (m_0)^t) = X^*_t+1(\mu) \cdot \mathbb{E}^{I''}(\bar{\tau}(I'') = t+1 | (m_0)^t). \]

Thus,
\[ X^*_t+1(\mu) = \mathbb{E}^{I''}(m_{t+1} = m_\mu | (m_0)^t). \]
These together imply

\[
U''(m_0^t) = \sum_{s=t+1}^{\infty} \sum_{\mu \in \Delta(\Theta)} \mathbb{P}''(m_s = m_\mu \mid (m_0)^t) \nu^*(\mu, s) \quad \text{(Lemma 10)}
\]

\[
= \sum_{s=t+1}^{\infty} \left( \sum_{\mu \in \Delta(\Theta)} \mathbb{P}''(m_s = m_\mu \mid (m_0)^{s-1}) \nu^*(\mu, s) \right) \mathbb{P}''((m_0)^{s-1} \mid (m_0)^t)
\]

\[
= \sum_{s=t}^{\infty} \left( \sum_{\mu \in \Delta(\Theta)} X_s^*(\mu) \nu^*(\mu, s) \right) \mathbb{P}''((m_0)^{s-1} \mid (m_0)^t)
\]

\[
= \sum_{s=t}^{\infty} \sum_{\mu \in \Delta(\Theta)} \mathbb{P}''(m_s = m_\mu, \tau(t') = s \mid (m_0)^t) \nu^*(\mu, s)
\]

\[
= U''(m_0^t).
\]

From Lemma 10, we know that the posterior beliefs of the history \((m_0)^t\) are the same under \(I'\) and \(I''\), which implies \(U''((m_0)^t) = U''((m_0)^t)\). Therefore, \(U''((m_0)^t) = U''((m_0)^t) \geq U''((m_0)^t) = U''((m_0)^t)\), which concludes the proof. \(\square\)

Finally, we conclude the proof of Proposition 5 by showing that the joint distributions of DM’s stopping times and actions under \(I\) and \(I''\) are identical.

**Proof of Proposition 5.** We showed in Step 1 that \(d(I, y) = d(I', y)\). By Lemma 12, we know that \(\tau(I'') = \tau(I'')\) equal in distribution. We showed in Lemma 12 that \(\tau(I') = \tau(I')\) equal in distribution, which implies \(\tau(I'') = \tau(I'')\) equal in distribution. Fix \(t \in T\) and an action \(a \in A\), because \((X_t, y_t)\) and \((X_t^*, y_t^*)\) induce the same distribution of actions under \(\nu_t\), we have

\[
\mathbb{P}''[a_{\tau(I'')} = a \mid \tau(I'') = t] = \sum_{\mu \in \Delta(\Theta)} X_t(\mu) y_t[\mu](a)
\]

\[
= \sum_{\mu \in \Delta(\Theta)} X_t^*(\mu) y_t^*[\mu^*](a)
\]

\[
= \mathbb{P}''[a_{\tau(I'')} = a \mid \tau(I'') = t].
\]

These together imply \(d(I, y) = d(I', y) = d(I'', y^*)\), as desired. \(\square\)
Appendix C: Details on topological space of information structures

We consider a DM’s utility function that can be written as $v(a, \theta, t) = u(a, \theta) - ct$. It is easy to see that the sets of sufficient stopping beliefs under DM’s static utility functions $v_t := v(\cdot, \cdot, t)$ are the same for all $t \in T$. Any extremal and obedient information structure $I$ induces a sufficient stopping belief and a joint distribution of stopping time over $\Delta(\Delta^*(\Theta) \times T)$. Let $\mathcal{D}$ be the set of feasible joint distribution of stopping time and sufficient stopping belief over $\Delta(\Delta^*(\Theta) \times T)$. We introduce a metric $\Delta$ of $\mathcal{D}$ as follows: for any $d_1, d_2 \in \mathcal{D}$,

$$\Delta(d_1, d_2) := \sum_{t \in T} \sum_{\mu \in \Delta^*(\Theta)} |d_1(\mu, t) - d_2(\mu, t)|c(t).$$

This metric is well-defined because the obedience constraint at $t = 0$ implies

$$\sum_{t \in T} \sum_{\mu \in \Delta^*(\Theta)} |d_1(\mu, t) - d_2(\mu, t)|c(t) \leq \sum_{t=0}^{\infty} d_1(t)c(t) + \sum_{t=0}^{\infty} d_2(t)c(t) \leq 2\phi^* < \infty.$$

It is easy to verify that this is indeed a metric.

**Proposition 6.** $\mathcal{D}$ is compact under the metric $\Delta$.

**Proof.** The proof is almost the same as Proposition 12 in Koh and Sanguanmoo (2022).

This proposition implies the existence of a solution to the optimization problem for the sender under the regularity assumption, and the set of the sender’s optimal information structure, says $\mathcal{D}^*$, is compact.

**Corollary 1.** When $|A| = 2$ and $|\Theta| = 2$, the optimization problem $\min_{d \in \mathcal{D}^*} \sum_{t \in T} d(\hat{\mu}, t)2^{-t}$ has a solution.

**Proof.** Define $f : \mathcal{D}^* \rightarrow \mathbb{R}^+$ as follows: $f(d) := \sum_{t \in T} d(\hat{\mu}, t)2^{-t}$. It is clear that $f$ is well-defined and continuous because, for every $d_1, d_2 \in \mathcal{D}$, $|f(d_1) - f(d_2)| \leq \Delta(d_1, d_2)/c$. This implies there exists $d^* \in \mathcal{D}^*$ such that $f(d^*) = \min_{d \in \mathcal{D}^*} f(d)$ because $\mathcal{D}^*$ is compact.