Dynamic Portfolio Choice with Intertemporal Hedging and Transaction Costs

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Abstract

When returns are partially predictable and trading is costly, CARA investors track a target portfolio at a constant trading speed. The target portfolio is optimal for a frictionless market, where asset returns are scaled back to account for trading costs and volatilities are adjusted to proxy execution risk. The trading speed solves an optimal execution problem, which describes how the legacy portfolio inherited from past trading is tilted towards the target portfolio in an optimal manner. Unlike for period-by-period mean-variance preferences, the target portfolio and trading speed are linked through a coupled system of Riccati equations, which describe how intertemporal hedging against changing investment opportunities [Merton 1971] interacts with the principle that one should “aim in front of the target” when trading is costly [Gärleanu and Pedersen 2013]. We illustrate the practical implications of these results for the model of Koijen et al. [2009], where return predictions are based on a short-term momentum and a long-term value signal.

JEL Classification: G11, G12.

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1 Introduction

Dynamic portfolio construction seeks to exploit changing investment opportunities whilst minimizing the costs of trading. The most basic paradigm in this context is diversification, often dubbed “the only free lunch in finance”. In the cross section, diversification goes back to Markowitz [1952] and already plays a central role in static settings. For dynamic portfolio choice, Merton [1971] first identified diversification over time as a crucial complementary principle. More specifically, he showed that if investment opportunities vary over time, then investors can hedge against these changes by holding securities whose returns are negatively correlated with changes in the investment opportunity set. The corresponding “intertemporal hedging portfolios” in turn reduce the long-run investment risk. Without trading costs, such optimal portfolios can be computed explicitly in a range of settings [Kim and Omberg 1996; Barberis 2000; Wachter 2002; Chacko and Viceira 2005; Liu 2007; Koijen et al. 2009; Basak and Chabakauri 2010; Guasoni and Robertson 2012].

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A different form of time diversification appears in the recent work of Gârleanu and Pedersen (2013, 2016a) and Collin-Dufresne et al. (2020). These authors use period-by-period mean-variance goal functionals that are “myopic” without transaction costs, in that each trade only takes into account current investment opportunities but not their future evolution or correlation with price shocks. Time diversification nevertheless comes into play when the costs of trading are taken into consideration. Indeed, when the optimal portfolio cannot be rebalanced freely at each time point, then each adjustment has to trade off current and future investment opportunities. Gârleanu and Pedersen (2013, 2016a) show that this leads to “aiming in front of the moving target”, i.e., tracking not the position that would maximize the risk-return tradeoff without transaction costs, but instead a weighted average of its current and future values.

In the present paper, we bring together these two different forms of time diversification. To wit, we start from the same quadratic transaction costs and linear state dynamics as in Gârleanu and Pedersen (2013, 2016a), but replace their period-by-period goal functional with a CARA utility function of future consumption streams. In this context, we then show that the investor still trades towards a target portfolio that is a weighted average of today’s and expected future desired portfolios, but these desired portfolios now hedge against adverse future changes in the investment opportunity set, as in Merton (1971).

The optimal target portfolio depends on the level of trading costs and therefore is not simply the solution of the frictionless version of the model. Instead, it can be interpreted as the optimizer of a frictionless portfolio problem where asset returns are shrunk to account for expected future trading costs, i.e., expressed on a net rather gross basis, and where the volatility of the assets is adjusted for execution risk. This suggests that investors can design their target portfolios as if there were no costs, if asset returns and risk are adjusted for trading costs in an appropriate manner. For example, if investment opportunities are persistent then the cost adjustment to returns is smaller than if investment opportunities are short lived. Similarly, if the investment opportunities are volatile then the adjustment to the execution risk is more significant than if investment opportunities are stable over time. The precise adjustment of the returns and volatility requires the complete solution to the optimization problem with trading costs, but this approach nevertheless accords closely with approaches adopted by portfolio managers. Indeed, these tend to scale back positions in assets or portfolios where the trading signals are short-lived, volatile, or expensive to trade.

The second ingredient for the optimal trading portfolio is the speed at which the target portfolio is tracked or, equivalently, undesirable “legacy positions” are traded away. In the model of Gârleanu and Pedersen (2013, 2016a) this trading speed can be computed independently from the target portfolio and only depends on a tradeoff between trading costs, risk aversion, and time discount rate, but not on the persistence or volatility of the predictive signals.

In contrast, with CARA preferences, the optimal trading speed is generally linked to the dynamics of the target position. Our model thereby allows us to assess in which situations the common practice to separate the design of portfolio allocations from the execution of trades is indeed justified.

Using dynamic programming arguments as in earlier versions of the present paper (Sefton and Champonnois, 2015) and recent independent work of Collin-Dufresne et al. (2022) (who consider more general preferences with source-dependent risk aversion), the solution of the full model with transaction costs can be expressed in terms of a system of matrix-valued Riccati equations. However, unlike in the local mean-variance model of Gârleanu and Pedersen (2013, 2016a) or in the

1For example, if the time discount rate is small and the trading cost matrix is proportional to the covariance matrix of price shocks, the optimal trading speed in Gârleanu and Pedersen (2016a, Proposition 1(iv)) is just the square root of risk aversion divided by trading costs, exactly the same “urgency parameter” that characterizes the optimal execution trajectory of a single exogenously given order in the model of Almgren and Chriss (2001).
frictionless case (Kim and Omberg, 1996), these equations are fully coupled and cannot be solved sequentially.

One major contribution of this paper is to give a clear interpretation to the structure of the optimal solution. To this end, we define a set of frictionless “shadow” prices which account for the impact of the optimal trades on the underlying price process. As markets are incomplete, the pricing kernel of these shadow prices is not unique and one of our Riccati equations calculates the corresponding minimax pricing kernel in the spirit of He and Pearson (1991), which equals the marginal utility of consumption along the optimal path. The target portfolio is the optimal dynamic portfolio for a frictionless problem where the actual prices are replaced by shadow prices. The trading rule is then to optimally trade away any “legacy positions”, the difference between actual portfolio holdings and the target portfolio. The corresponding optimal trading speed is the solution to an optimal execution problem, but where again the volatility of the asset returns is the volatility of the shadow prices rather than the actual prices. This trading speed is calculated in terms of the solution to another Riccati equation. The final, and third, Riccati equation couples these two other equations and calculates the price impact consistent with these two otherwise separate problems.

A significant technical contribution of the present paper is to prove both the existence of an appropriate solution to this coupled Riccati system, and to demonstrate how to solve it numerically in a straightforward and efficient manner even for many assets and trading signals. Systems of Riccati equations generically have many solutions and, with multiple assets and trading signals, and it can quickly become a cumbersome problem to identify the solution that indeed identifies the value function and optimal policy for the problem at hand. For a single asset and trading signal, the optimality equations can be reduced to a single nonlinear equation for which existence follows directly from the intermediate value theorem as in Collin-Dufresne et al. (2022), but such explicit calculations are no longer possible with multiple assets or trading signals.

However, we prove that by continuously extending the no-trade solution for infinitely large transaction costs, one can always construct a solution of the Riccati system. The latter in turn automatically has the required properties to carry out a rigorous verification argument. Unlike in the frictionless version of the model, no parameter constraints are needed to rule out “Nirvana solutions” (Kim and Omberg, 1996) – arbitrarily small but superlinear transaction costs guarantee that a solution always exists even if frictionless trading opportunities are excessively good.

In addition to establishing existence in this manner, we also show how to solve the Riccati system of equations using standard numerical linear algebra. As alluded to above, the challenge is that there are generally many different solutions. However, the correct one can be constructed directly from the eigenvalues and eigenvectors of a Hamiltonian matrix obtained by aggregating the Riccati system. More specifically, solving a linear system associated to the negative eigenvalues produces the solution with right stability properties needed to verify optimality of the corresponding policy. Both steps of this procedure can be easily implemented even in high dimensions using standard numerical algorithms with just a few lines of code.

To illustrate both the applicability of this method and investigate the interplay of transaction costs and changing investment opportunities in a realistic setting, we apply our general results to the model of Koijen et al. (2009). These authors study a frictionless model where investment opportunities arise from two time varying factors, namely short-run momentum and a value signal that describes mean reversion over longer time scales. Our model in turn allows to assess how the sizeable intertemporal hedging portfolios identified by Koijen et al. (2009) change when transaction costs after taken into consideration. In particular, one can assess to what extent intertemporal hedging against changing investment opportunities still plays an important role when trading is costly.
As illustrated in Figure 1, the momentum factor fluctuates much more rapidly than the value signal, which moves over much longer time scales. Smaller funds can exploit the short-term trading opportunities associated with momentum rather efficiently even when price impact costs are taken into account, see the left panel of Figure 1. In contrast, the trading costs of larger funds do not allow them to track these short-term signals in the same way and their portfolio allocation is in turn closely linked to the long-term value factor as illustrated in the right panel of Figure 1.

This clearly manifests itself in the corresponding intertemporal hedging portfolios: whereas the adjustments to momentum are quickly scaled back if trading costs are high or the fund is large, adjustments to value remain largely unchanged. In particular, the strong negative correlation between value and price shocks estimated by Koijen et al. (2009) allows to hold larger risky positions than in the mean-variance version of the model studied by Gärleanu and Pedersen (2016a). This effect is reversed for either smaller correlations between price and value shocks or for higher interest rates. In both cases, the value factor becomes less attractive, either because it is less useful as an intertemporal hedge or because the investor’s effective horizon shrinks so that long-term risks become less important. As a consequence, the average risky investment in our model can in turn drop below the mean-variance allocation of Gärleanu and Pedersen (2016a). The reason for this is that the short-term momentum signal then dominates, which is positively correlated with price shocks and in turn produces intertemporal hedging terms with the opposite sign.

Another important insight from the two-factor model is that even when the slow value signal is the main driver of the average portfolio holdings as in our baseline setting, the fast momentum factor still does not become irrelevant to how the portfolio is traded. Instead, the optimal trading speed in our model is about 25% higher than for Gärleanu and Pedersen (2016a), which is caused...
by the extra volatility in market expected returns caused by the momentum factor. In contrast, in a model with only a value signal as in Barberis (2000) or the numerical examples of Collin-Dufresne et al. (2022), the trading speed is smaller than the one of Gărleanu and Pedersen (2016a) as in the absence of a momentum factor, mean-reversion implies that the expected returns of the market are less volatile in the long run.

In summary, we see that even for large funds for which trading costs caused by price impact are a major concern, intertemporal hedging remains highly relevant. However, the corresponding portfolio adjustments are sensitive to transaction costs (if the underlying signals decay quickly or are volatility) and to interest rates (for persistent signals). The general toolbox developed in this paper conveniently allows to compute these adjustments in a systematic and effective manner.

The remainder of the this article is organized as follows. Section 2 introduces our lifetime consumption model with partially predictable returns and transaction costs. Our main results on the optimal portfolio/consumption policy in this context are presented in Section 3. Subsequently, Section 4 applies these general results to the value/momentum model of Koijen et al. (2009). Section 5 concludes. For better readability, all proofs are collected in the appendices.

2 Model

Throughout, we fix a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) supporting a \(k\)-dimensional standard Brownian motion \(\{B(t)\}_{t \geq 0}\).

2.1 Financial Market

We consider a financial market with \(1+m\) assets. The first one is safe and earns a constant interest rate \(r > 0\). The other \(m\) assets are risky, with dynamics

\[
\begin{align*}
dP(t) &= (\bar{\mu} + C_X X(t) + rP(t)) \, dt + \sigma_p dB(t), \\
P(0) &= p_0 \in \mathbb{R}^m.
\end{align*}
\]

Here, the volatility matrix \(\sigma_p \in \mathbb{R}^{m \times k}\) of price shocks is deterministic and the corresponding covariance matrix \(\sigma_p \sigma_p^\top\) is invertible. The expected excess returns are composed of a constant \(\bar{\mu} \in \mathbb{R}^m\) and a mean-reverting component, which is the product of a scaling matrix \(C_x \in \mathbb{R}^{m \times d}\) and an \(\mathbb{R}^d\)-valued Ornstein-Uhlenbeck process:

\[
\begin{align*}
dX(t) &= A_x X(t) \, dt + \sigma_x dB(t), \\
X(0) &= x_0 \in \mathbb{R}^d.
\end{align*}
\]

Here, \(\sigma_x \in \mathbb{R}^{d \times k}\) and the corresponding covariance matrix \(\sigma_x\sigma_x^\top\) of shocks to investment opportunities is invertible. The matrix \(A_x\) is stable (i.e., all real parts of its eigenvalues are negative), so that \(X(t)\) is indeed mean reverting around zero and the expected excess returns of the risky assets in turn fluctuate around \(\bar{\mu}\).

2.2 Transaction Costs and Goal Functionals

As in Gărleanu and Pedersen (2013, 2016a), trades in the risky assets incur quadratic costs \(\frac{1}{2} \dot{\theta}(t)^\top \Lambda \dot{\theta}(t)\) for a symmetric, positive definite matrix \(\Lambda \in \mathbb{R}^{m \times m}\), levied on the rates with which the risky positions are adjusted:

\[
\begin{align*}
d\theta(t) &= \dot{\theta}(t) dt, \\
\theta(0) &= \theta_0 \in \mathbb{R}^m.
\end{align*}
\]
The wealth dynamics of an agent who trades at rate \( \dot{\theta}(t) \) and consumes at rate \( c(t) \) in turn are

\[
dW^c \dot{\theta}(t) = \left( rW^c \dot{\theta}(t) - c(t) + \theta(t)^\top (\bar{\mu} + C_x X(t)) - \frac{1}{2} \dot{\theta}(t)^\top \Lambda \dot{\theta}(t) \right) dt + \theta(t)^\top \sigma_p dB(t),
\]

\( W^c \dot{\theta}(0) = w_0 \in \mathbb{R}. \)

For agents with constant absolute risk aversion \( \beta > 0 \) and time-discount rate \( \delta > 0 \), we consider the standard problem of maximizing lifetime utility from consumption:

\[
\mathbb{E} \left[ \int_0^\infty e^{-\delta t - \beta c(t)} dt \right] \to \max!
\]

Here, in order to rule out doubling strategies and excessive borrowing, the risky positions \( \theta(t) \) and the wealth \( W^c \dot{\theta}(t) \) corresponding to the admissible consumption and trading rates \( c(t), \dot{\theta}(t) \) need to satisfy suitable integrability and transversality conditions, see Appendix D.

### 3 Main Results

Our main result identifies the optimal trading and consumption rates for (2.4) in terms of a coupled system of matrix Riccati equations. As in the recent work of Collin-Dufresne et al. (2022) or earlier versions of the present paper Sefton and Champonnois (2015), these equations can be derived heuristically using standard dynamic programming arguments outlined in Appendix A. However, unlike for the mean-variance model of Garleanu and Pedersen (2013, 2016a), these equations are fully coupled so that the existence of a solution can not be inferred from explicit computations. Moreover, matrix Riccati equations generically have many solutions, so that it is a delicate question – in particular for multiple assets and state variables – to identify the correct one that indeed leads to the value function and optimal policy for the problem at hand.

In the present paper, we overcome these difficulties. More specifically, by continuously extending the explicit solution for arbitrarily large transaction costs, we construct a solution of the Riccati system in Appendices B and C. In Appendix D we then prove a rigorous verification theorem that shows that our solution indeed has the required properties to identify the value function and the optimal policies. These arguments do not require any smallness conditions or other assumptions on the model parameters. In particular, “Nirvana solutions” as in Kim and Omberg (1996) never exist with superlinear transaction costs.

**Theorem 3.1.** Set \( \gamma = r\beta \).

(i) There exists a unique solution \( \Pi_{\theta \theta}, \Pi_{xx}, \Pi_{\theta x} \) of the matrix Riccati system

\[
\Pi_{\theta \theta} \Lambda^{-1} \Pi_{\theta \theta} + r \Pi_{\theta \theta} - \gamma (\Pi_{\theta x} \sigma_x - \sigma_p) (\Pi_{\theta x} \sigma_x - \sigma_p)^\top = 0, \tag{3.1}
\]

\[
\left( A_x - \frac{r}{2} I_d \right)^\top \Pi_{xx} + \Pi_{xx} \left( A_x - \frac{r}{2} I_d \right) - \gamma \Pi_{xx} \sigma_x \sigma_x^\top \Pi_{xx} + \Pi_{\theta x} \Lambda^{-1} \Pi_{\theta x} = 0, \tag{3.2}
\]

\[
- \left( \Lambda^{-1} \Pi_{\theta \theta} + \frac{r}{2} I_m \right)^\top \Pi_{\theta x} + \Pi_{\theta x} \left( A_x - \frac{r}{2} I_d - \gamma \sigma_x \sigma_x^\top \Pi_{xx} \right) + \gamma \sigma_p \sigma_x^\top \Pi_{xx} - C_x = 0, \tag{3.3}
\]

for which \( \Pi_{\theta \theta} \) and \( \Pi_{xx} \) are positive semidefinite.

(ii) There exists a unique solution \( \Pi_{\theta}, \Pi_{x} \) of the linear system

\[
\gamma (\Pi_{\theta x} \sigma_x - \sigma_p) \sigma_x^\top \Pi_{x} - (\Pi_{\theta \theta} \Lambda^{-1} + r I_m) \Pi_{\theta} = \bar{\mu}, \tag{3.4}
\]

\[
\left( A_x - r I_d - \gamma \sigma_x \sigma_x^\top \Pi_{xx} \right)^\top \Pi_{x} - \Pi_{\theta x} \Lambda^{-1} \Pi_{\theta} = 0. \tag{3.5}
\]
For
\[ J_0 = \frac{1}{r} \exp \left( \frac{1}{r} \left( r - \delta - \frac{\gamma}{2} \Pi_1 \Lambda^{-1} \Pi_0 + \frac{\gamma^2}{2} \Pi_x \sigma_x \sigma_x^\top \Pi_x - \frac{\gamma}{2} \text{tr} \left( \sigma_x^\top \Pi_{xx} \sigma_x \right) \right) \right), \]  
(3.6)

define
\[ J(x, w, \theta) = -J_0 \exp \left( -\gamma \left( w + \frac{1}{2} x^\top \Pi_{xx} x - \Pi_x^\top x - \frac{1}{2} \theta^\top \Pi_{\theta\theta} \theta - \Pi_{\theta x}^\top \theta \right) \right), \]  
(3.7)

Then, for any admissible investment/consumption policy,
\[ J(x_0, w_0, \theta_0) \geq \mathbb{E} \left[ \int_0^\infty -e^{-\delta t - \beta c(t)} \, dt \right], \]  
and this upper bound is attained by the feedback policy
\[ c(t) = \frac{r}{\gamma} \log \left( -r J(X(t), W_{c}, \dot{\theta}(t), \theta(t)) \right), \]  
(3.8)
\[ \dot{\theta}(t) = -\Lambda^{-1} \left( \Pi_{\theta\theta} \theta(t) + \Pi_\theta + \Pi_{\theta x} X(t) \right). \]  
(3.9)

Therefore, this investment/consumption policy is optimal for the lifetime consumption problem (2.4) and \( J \) is the corresponding value function.

3.1 Numerical Solution of the Riccati System

Riccati systems generally have many solutions and identifying the one with the correct signs quickly becomes a challenging task for multiple assets and trading signals. We therefore now outline how to systematically solve this problem using the eigenvalues and eigenvectors of the Hamiltonian matrix
\[ H = \begin{bmatrix} A & R \\ -Q & -A^\top \end{bmatrix}, \]  
(3.10)

where
\[ R = \begin{bmatrix} -\gamma \sigma_x \sigma_x^\top & 0 \\ 0 & \Lambda^{-1} \end{bmatrix}, \quad A = \begin{bmatrix} A_x - \frac{\gamma}{2} I_d \\ 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & C_x \\ C_x & -\gamma \sigma_p \sigma_p^\top \end{bmatrix}. \]

This computation can be implemented simply and efficiently even in high dimensions using standard numerical algebra packages. Note that \( H \) is similar to \(-H^\top\), hence if \( h \in \mathbb{C} \) is an eigenvalue of \( H \), then so is \(-h^*\). Therefore, if we focus on the generic case where \( H \) has no purely imaginary eigenvalues, then there are \( d + m \) eigenvalues with negative real parts and \( d + m \) eigenvalues with positive real parts.

To construct the relevant solution of the Riccati system (3.1)-(3.3), we focus on the \( d + m \) eigenvectors corresponding to the eigenvalues with negative real parts and define the matrix \([\Pi_1^\top, \Pi_2^\top]^\top\) with \( \Pi_1, \Pi_2 \in \mathbb{R}^{(d+m) \times (d+m)} \) obtained by stacking up these eigenvectors. In the generic case where \( \Pi_1 \) is invertible (which can be checked by computing its singular value decomposition, for example), we then set \( \Pi := \Pi_2 \Pi_1^{-1} \), which can be computed by solving a system of linear equations. This construction then indeed leads to a solution of the Riccati system (3.1)-(3.3). To see this, multiply the Hamiltonian matrix \( H \) from the left by the matrix \([\Pi, -I_d] \) and from the right by the matrix \([I_{d+m}, \Pi]^\top = [\Pi_1^\top, \Pi_2^\top] \Pi_1^{-1} \) and recall that the columns of \([\Pi_1^\top, \Pi_2^\top]^\top \) are the eigenvectors of \( H \).

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Writing $D^H$ for the diagonal matrix populated by the corresponding eigenvalues, it then follows that

$$\Pi R\Pi + \Pi A + A^\top \Pi + Q = \begin{bmatrix} \Pi, -I_{d+m} \end{bmatrix} H \begin{bmatrix} I_{d+m} \\ \Pi \end{bmatrix} = \begin{bmatrix} \Pi, -I_{d+m} \end{bmatrix} I_{d+m} \Pi_1 D^H \Pi_1^{-1} = 0.$$  

The diagonal elements of $\Pi R\Pi + \Pi A + A^\top \Pi + Q$ correspond to the left-hand side of the Riccati equations (3.1) and (3.2) respectively; the off-diagonal elements correspond to the left-hand side of (3.3). Whence, $\Pi_2 \Pi_1^{-1}$ indeed is a solution of (3.1)-(3.3). Moreover, by [Zhou et al. 1996, Theorem 13.5], this is the unique solution such that $(A + R\Pi)$ is stable. We show in the proof of Theorem 3.1 that the solution we construct there and link to the value function of our problem has the same stability property, cf. Lemma B.3. As a consequence, the two solutions coincide:

$$\Pi = \begin{bmatrix} \Pi_{xx} & -\Pi_{\theta x}^\top \\ -\Pi_{\theta x} & -\Pi_{\theta \theta} \end{bmatrix},$$

As a consequence, determining the relevant solution $\Pi_{xx}$, $\Pi_{\theta x}$, $\Pi_{\theta \theta}$ of the Riccati system therefore boils down to (i) computing the eigenvalues and eigenvectors of the Hamiltonian matrix $H$ and then (ii) solving a system of linear equations. Once $\Pi_{xx}$, $\Pi_{\theta x}$, $\Pi_{\theta \theta}$ are computed in this manner, the remaining coefficients $\Pi_x$, $\Pi_\theta$ can be readily computed by solving some further systems of linear equations. All of these steps can be conveniently carried out using standard numerical linear algebra packages with just a few lines of code and easily scale to high dimensions.

### 3.2 Interpretation of the Solution

After discussing their computation, we now turn to the interpretation of the optimal consumption $c(t)$ from (3.8) and the optimal trading rate $\dot{\theta}(t)$ from (3.9).

**Pricing Kernel and the Consumption Stream** The marginal utility of the optimal consumption plays a central role in classical frictionless asset pricing, because it singles out the pricing kernel that the investor at hand uses to price future random cash flows. We now outline how this link extends to the present setting with transaction costs. To this end, note that the marginal utility of consumption is

$$\phi(t) = \beta e^{-\beta c(t) - \delta t} = e^{-\delta t} \frac{\partial J}{\partial w},$$

where the second equality follows immediately from the first-order condition (A.2). Using the three Riccati equations (3.1), (3.2) and (3.3), it can be shown that the marginal utility has dynamics

$$\frac{d\phi(t)}{\phi(t)} = -rdt - \gamma \left( \theta^\top(t)(\sigma_p - \Pi_{\theta x}\sigma_x) + (\Pi_{xx}X(t) - \Pi_x)^\top \sigma_x \right) dB(t). \quad (3.11)$$

Whence, just like a standard frictionless pricing kernel, the drift rate of the marginal utility of consumption along the optimal path is pinned down by the risk free rate. However, with transaction costs, this pricing kernel does not price the unaffected price $P(t)$, but instead the trade prices $\overline{P}(t) = P(t) + \Lambda \dot{\theta}(t)$ at which the transactions of the optimal policy are executed. To wit, the process $\{\phi(t)\overline{P}(t)\}_{t \geq 0}$ is a martingale (see Appendix E.1), so that

$$\overline{P}(t) = \mathbb{E}_t \left[ \frac{\phi(T)}{\phi(t)} \overline{P}(T) \right], \quad \text{for } 0 \leq t \leq T,$$

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in analogy to the standard frictionless asset pricing theory. This suggests that the trade price $\bar{P}(t)$ rather than the unaffected price $P(t)$ is the natural frictionless reference in the present context. We now provide an interpretation for the optimal trading strategy from Theorem 3.1 based on this observation.

**Target Portfolio as a Frictionless Optimizer** To simplify the exposition, we assume that the matrix $\Pi_{\theta\theta}$ is invertible. Then, the optimal trading rate (3.9) can be written as

$$\dot{\theta}(t) = -\Lambda^{-1} \Pi_{\theta\theta} \left( \theta(t) - \hat{\theta}(X(t)) \right).$$

Put differently, it is optimal to trade with a constant trading speed of $\Lambda^{-1} \Pi_{\theta\theta}$ from the current holdings $\theta(t)$ towards the target portfolio

$$\hat{\theta}(X(t)) = -\Pi_{\theta\theta}^{-1} (\Pi_{\theta x} X(t) + \Pi_{\theta}).$$

Equivalently, the excess holdings $\theta(t) - \hat{\theta}^*(X(t))$ relative to the target that have been inherited from previous trading (the legacy portfolio) are traded away at the constant rate $\Lambda^{-1} \Pi_{\theta\theta}$.

Like the optimal portfolio in the frictionless version of the model, the target portfolio is a function of the signal state $X(t)$ only, but does not depend on the current portfolio holdings $\theta(t)$. This motivates the conjecture that the target portfolio is a frictionless optimizer, if the price dynamics are adjusted for transaction costs in a suitable manner.

In view of its link to the marginal-utility based pricing kernel established in the previous section, the trade prices $\bar{P}(t)$ are the natural candidate for this relationship. To investigate, this, we first compute their dynamics:

$$d\bar{P}(t) = dP(t) - \left( \Pi_{\theta\theta} \dot{\theta}(t) dt + \Pi_{\theta x} dX(t) \right)$$

$$= \left( \hat{\mu} + \hat{C}_x X(t) + r \bar{P}(t) \right) dt + \hat{\sigma}_p dB(t) + \gamma \hat{\sigma}_p \hat{\sigma}_p^\top \left( \theta(t) - \hat{\theta}^*(X(t)) \right) dt. \quad (3.13)$$

Here, in the second line, we substituted in for the differentials from Equations (2.1), (2.2) and (3.9), simplified using the Riccati Equation (3.1), and introduced the notation

$$\hat{C}_x = C_x - \Pi_{\theta x} A_x, \quad \hat{\sigma}_p = \sigma_p - \Pi_{\theta x} \sigma_x. \quad (3.15)$$

In view of (3.14), the dynamics of the trade price therefore can be split into two components. The dynamics of the first component only depend on the signal states,

$$d\bar{P}(t) = \left( \hat{\mu} + \hat{C}_x X(t) + r \bar{P}(t) \right) dt + \hat{\sigma}_p dB(t). \quad (3.16)$$

In contrast, the dynamics of the second component are fully determined by the portfolio’s legacy holdings $\theta(t) - \hat{\theta}^*(X(t))$. The state-dependent part of the trade prices then indeed is a frictionless “shadow price”, for which the optimal portfolio coincides with the target portfolio (3.12). We can emphasise this connection by rewriting the optimal target portfolio by substituting (3.1) and (3.3) into (3.12):

$$\hat{\theta}^*(x) = \frac{1}{\gamma} \left( \hat{\sigma}_p \hat{\sigma}_p^\top \right)^{-1} \left( \hat{\mu} + \hat{C}_x x - \gamma \hat{\sigma}_p \hat{\sigma}_p^\top (\Pi_{xx} x - \Pi_x) \right). \quad (3.17)$$

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2Similar relationships between trade prices and pricing kernels appear in the works of Dolinsky and Soner (2013); Guasoni and Rásonyi (2015) on general abstract models with superlinear transaction costs.

3Sufficient conditions for this are provided in Appendix E.2.
The first two terms on the right-hand side are the myopic mean-variance portfolio but with respect to the shadow prices rather than actual prices. The last term is the sum of \( d \) hedging portfolios, whose shadow innovations are maximally correlated with innovation to the state process; the portfolio weights are \((\tilde{\sigma}_p \tilde{\sigma}_x)\)^{-1} \( \tilde{\sigma}_p \sigma_x \). The final hedging portfolio is a weighted sum of these \( d \) portfolios, where the weights are the price of the respective state risk, see Equation (3.11).

The following theorem shows that the optimal consumption stream of the embedded frictionless problem also matches that of the frictional problem if transaction costs along the optimal path are accounted for by a suitable “trading levy” imposed on the initial endowment and the wealth dynamics:

**Theorem 3.2.** For the frictionless price process \((3.16)\), consider the lifetime portfolio/consumption problem of maximizing \( \mathbb{E}[\int_0^\infty -e^{-\delta t - \beta t} dt] \) without transaction costs. Then, the target portfolio \( \tilde{\theta}(t) \) is the optimal trading strategy for this problem. Moreover, if we define the modified frictionless wealth process

\[
d\tilde{W}(t) = \left[ r\tilde{W}(t) - \tilde{\gamma}(t) - r\tilde{y}(X(t)) \right] dt + \tilde{\theta}(t) \left[ \left( \hat{\mu} + \tilde{C}_x X(t) \right) dt + \tilde{\sigma}_p dB(t) \right], \quad \tilde{W}(0) = w_0 + \tilde{y}(x),
\]

where

\[
\tilde{y}(x) = \frac{1}{2} (\Pi_{\theta x} x + \Pi_{\theta})^\top \Pi_{\theta}^{-1} (\Pi_{\theta x} x + \Pi_{\theta}),
\]

then the optimal consumption is

\[
\tilde{c} = c + \frac{\gamma}{2} \left( \theta - \tilde{\theta}^*(x) \right)^\top \Pi_{\theta \theta} \left( \theta - \tilde{\theta}^*(x) \right),
\]

and the corresponding frictionless value function satisfies

\[
J(x, \tilde{w}) = \tilde{J}(x, \tilde{w}) \exp \left( \frac{\gamma}{2} \left( \theta - \tilde{\theta}^*(x) \right)^\top \Pi_{\theta \theta} \left( \theta - \tilde{\theta}^*(x) \right) \right) \leq \tilde{J}(x, \tilde{w}) \leq 0. \tag{3.20}
\]

Theorem 3.2 states that the frictionless optimal portfolio for the adjusted price process price process \((3.16)\) is precisely the target portfolio \((3.17)\) from the problem with transaction costs. Moreover, if the initial portfolio coincides with the target portfolio, then the value functions and optimal consumptions for the original problem with transaction costs and for the frictionless shadow problem also coincide. To achieve this matching, both the drift and diffusion terms of the shadow price process, \( \tilde{P}(t) \) are adjusted for trading costs relative to the actual price process \( P(t) \). To understand these adjustments, observe that Equation (3.16) implies \( \tilde{P}(t) = P(t) - \Pi_{\theta x} X(t) \). Whence, the term \(-\Pi_{\theta x}\) describes the impact of transaction costs attributable to changes in the target portfolio, distinct from the cost impact attributable to trading down the legacy portfolio in Equation (3.13). This cost impact affects both the expected return and the volatility of shadow asset returns, \( d\tilde{P}(t) \). The impact on expected returns is \(-\Pi_{\theta x} A_x X(t) \). This is the price impact due to trading out of portfolio positions as the signal decays. The other impact cost adjustment is to volatility, \( \tilde{\sigma}_p = \sigma_p - \Pi_{\theta x} \sigma_x \). Our investors are risk-averse to the extra trading costs resulting from stochastic changes to the investment opportunity set. Thus if the long-run volatility of the asset return is greater than in the short run, then this adjustment will increase the volatility of that asset’s returns; if it is less, it will decrease it.\(^5\)

\(^4\)These weights can be understood as the coefficients of a regression of the asset innovations on the state innovations.

\(^5\)In our example later, the momentum factor increases the long run volatility of the market returns, whereas the mean-reversion factor decreases it. Thus two effects are countervailing. The net effect will depend on the investment horizon, or discount rate. In our base case the momentum factor dominates.
If the initial portfolio does not coincide with the target portfolio, then the consumption stream in the frictional problem will be lower by an adjustment, \( \frac{1}{2} (\theta - \hat{\theta}(x))^\top \Pi_{\theta\theta} (\theta - \hat{\theta}(x)) \) to account for the costs of trading out of this initial legacy position. Similarly, the value function will also be lower. We now link these adjustment to an optimal execution problem, which also provides a natural interpretation for the optimal trading rate.

**Trading Speed Determined by Optimal Execution** More specifically, we now show that the optimal trading speed \( \Lambda^{-1} \Pi_{\theta\theta} \) arises from the solution of an “optimal execution problem”, where the risky assets offer no risk premium, so that it is optimal to trade out of the initial positions in a way that balances trading costs and risk appropriately. The solution to this problem will also offer an interpretation of the correction terms for legacy holdings and transaction costs in Theorem 3.2. The same reasoning also explains the adjustment terms in Theorem 3.2. To wit, in Equation (3.21) of Theorem 3.2, the difference in the value function of the frictional problem relative to the frictionless is a direct result of an effective reduction in wealth by the total cost of trading out any initial legacy positions. Similarly in (3.19), the consumption stream of the frictional problem relative to the frictionless is reduced by the instantaneous charge of exiting these initial legacy positions. In addition, we can explain the levy cost, \( \hat{y}(X(t)) \) in Theorem 3.2. The optimal portfolio to the frictionless problem is the target portfolio \( \hat{\theta}^*(x) = -\Pi_{\theta\theta}^{-1} (\Pi_{\theta x} + \Pi_{\theta}) \). However, given the initial states values are expected to decay over time and in the very long run tend to 0, it will be necessary to exit this initial target portfolio. To match the consumption streams of the frictional with that of the frictionless problem, it is necessary to adjust for the costs of exiting these positions. The adjustment required is to increase the initial wealth at \( t = 0 \) by the total cost of exiting this target portfolio, \( \hat{y} = \frac{1}{2} \hat{\theta}^*(x) \Pi_{\theta\theta} \hat{\theta}^*(x) \). Over time, a levy is then charged against this wealth, \( r \hat{y} \), etc.
equal to the expected cost of trading out of these positions at any point in time. At time \( t = 0 \), the income from the wealth adjustment and the levy charge exactly offset each other. However, this wealth is then run down in line with expected trading costs.

On the second point, the trading rate in the full frictional problem is the trading rate in the optimal execution problem. More specifically, the trading rate \( \Lambda^{-1}\Pi_{\theta\theta} \) has the explicit expression

\[
\Lambda^{-1}\Pi_{\theta\theta} = \Lambda^{-1/2} \left( \gamma \Lambda^{-1/2} \hat{\sigma}_p \hat{\sigma}_p^\top \Lambda^{-1/2} + \frac{r^2}{4} \right)^{1/2} \Lambda^{1/2} - \frac{r}{2}.
\]  

(3.22)

This parallels the optimal trading rate in (Gărleanu and Pedersen 2016a, Proposition 1) except that the price volatility \( \sigma_p \) in their Formula (10) has been replaced by the price volatility of shadow prices, \( \hat{\sigma}_p = \sigma_p - \Pi_{\theta\theta} \sigma_x \) which includes an adjustment for execution risk. We explore this link in more detail next.

### 3.3 Links to the Linear-Quadratic Model of Gărleanu and Pedersen

For linear state dynamics (Gărleanu and Pedersen 2016a, Assumption A.1), their price and state process also follow (2.2) and (2.1) respectively\(^6\). However, their agent maximises a linear-quadratic (LQ) objective\(^7\):

\[
\max_{\hat{\theta}} \mathbb{E} \left[ \int_0^\infty e^{-rt} \left( \theta(u)^\top C_x X(u) - \frac{\gamma}{2} \left\| \sigma_p^\top \theta(u) \right\|^2 - \frac{1}{2} \hat{\theta}(u)^\top \Lambda \hat{\theta}(u) \right) du \right].
\]

(3.23)

Theil (1957) first outlined the “certainty equivalence property” of LQ controllers. This means that an agent maximising an LQ objective in the face of future uncertainty proceeds “as if” the uncertain elements were equal to their expected values. In the framework of Gărleanu and Pedersen (2016a), this implies that the agents proceed “as if” the trading signal was deterministic with signal volatility \( \sigma_x = 0 \). In particular, there is no hedging against future changes in the investment opportunity set in this case.

If we set \( \sigma_x = 0 \) in our three Riccati equations (3.1), (3.2) and (3.3), we obtain the three corresponding equations (A.2), (A.14) and (A.15) of Gărleanu and Pedersen (2016b), up to a change of notation\(^8\). The Riccati equations then are no longer coupled, because \( \sigma_x = 0 \) implies \( \sigma_p = \hat{\sigma}_p \), and so \( \Pi_{\theta\theta} \) is explicitly given by (3.22) just like in Gărleanu and Pedersen (2016a). The target portfolio then coincides as well.

For general signal volatilities \( \sigma_x \), the optimal trading rate (3.9) for our lifetime consumption problem (2.4) also maximizes a mean-variance goal functional, but only if the risk term is modified appropriately:

**Lemma 3.4.** The feedback trading rate \( \hat{\theta}(t) \) from (3.9) is also optimal for the following LQ problem:

\[
\max_{\hat{\theta}} \mathbb{E} \left[ \int_0^\infty e^{-rt} \left( \theta(u)^\top \left( \hat{\mu} + C_x X(u) \right) - \frac{\gamma}{2} \left\| \hat{\sigma}_p^\top \theta(u) + \sigma_x^\top \left( \Pi_{xx} X(u) - \Pi_x \right) \right\|^2 - \frac{1}{2} \hat{\theta}(u)^\top \Lambda \hat{\theta}(u) \right) du \right].
\]

---

\(^6\)The only difference is that they do not include a constant return \( \overline{\mu} \) in their specification; then, the constants \( \Pi_x \) and \( \Pi_{xx} \) vanish.

\(^7\)To be precise, Gărleanu and Pedersen (2016a) use a different discount rate \( \rho \) rather than \( r \), the return to savings, in their objective function. However, their optimal investment strategy is only function of \( \rho \) with \( r \) not playing any role. Given this, we simplify and remove the distinction in our discussion by setting \( \rho = r \).

\(^8\)To wit, their matrices \( B \) and \( -\Phi \) correspond to our \( C_x \) and \( A_x \). Moreover, their coefficients \( A_{ff}, A_{xf} \) and \( A_{xx} \) correspond to our \( \Pi_{xx}, \Pi_{\theta x} \) and \( \Pi_{\theta\theta} \), respectively.
We see that expected returns and transaction costs in this LQ problem remain unchanged relative to our original lifetime consumption problem. The difference is how risk is accounted for. When \(\sigma_x = 0\), only the infinitesimal portfolio variance appears as in Garleanu and Pedersen (2013, 2016a). But when trading signals fluctuate randomly, the price volatility is adjusted according to (3.15) and there is an additional state dependent risk term \(-\sigma_p^\top (\Pi_{xx} X(u) - \Pi_x)\). To understand the significance of this term, note that the pricing kernel from (3.11) can be rewritten as

\[
\frac{d\phi(t)}{\phi(t)} = -rdt - (\hat{\mu} + \hat{C}_x X(t))^\top \left( \tilde{\sigma}_p \tilde{\sigma}_p^\top \right)^{-1} \tilde{\sigma}_p dB(t)
\]

\[
- \gamma (\Pi_{xx} X(t) - \Pi_x)^\top \sigma_x \left( I - \tilde{\sigma}_p^\top \left( \tilde{\sigma}_p \tilde{\sigma}_p^\top \right)^{-1} \tilde{\sigma}_p \right) dB(t)
\]

\[
- \gamma \left( \theta(t) - \hat{\theta}^*(X(t)) \right)^\top \tilde{\sigma}_p dB(t).
\]

Here, the first term prices shocks to the shadow price, whereas the second and third price orthogonal state shocks and legacy holdings, respectively. We see that the market price of state shocks orthogonal to prices is precisely the extra risk term that appears in the mean variance representation from Lemma 3.4. In particular, the corresponding risk term depends both the infinitesimal variance of the trade prices and the orthogonal shocks to the trading signals.

4 Example: The Momentum/Value Model of Koijen et al. (2009)

As a concrete example for both our theoretical results and our numerical solver, we now focus on the momentum/value model proposed and analyzed in a frictionless setting by Koijen et al. (2009).

4.1 Model

In Koijen et al. (2009), expected returns are predicted by two signals. The first is an exogenous “value factor” \(V(t)\) modelled by an Ornstein-Uhlenbeck process; it is fit to dividend yields in the empirical part of their paper. The second “momentum” signal \(M(t)\) driving future expected returns is an exponentially weighted moving average of the realized stock returns themselves. This process again has linear mean-reverting dynamics, but now the stochastic mean-reversion level is determined by the value factor.

The joint dynamics of the two signal processes is a multivariate Ornstein-Uhlenbeck process in line with (2.2):

\[
\frac{dX_t}{X(t)} = \begin{bmatrix}
(1 - \phi)(1 - \phi) \mu_1 s \\
0 \\
-\alpha
\end{bmatrix}
\begin{bmatrix}
M(t) \\
V(t)/s
\end{bmatrix}
\begin{bmatrix}
\sigma_S \\
\sigma_{X_1}/s \\
\sigma_{X_2}/s
\end{bmatrix}
\frac{dB(t)}{X(t)} + \begin{bmatrix}
\sigma_{S} \\
\sigma_{X_1}/s \\
\sigma_{X_2}/s
\end{bmatrix}
\frac{dB(t)}{X(t)} + \begin{bmatrix}
m_0 \\
v_0/s
\end{bmatrix}.
\]

\[\frac{dM(t)}{V(t)/s} = \begin{bmatrix}
(1 - \phi) \\
0 \\
-\alpha
\end{bmatrix}
\begin{bmatrix}
M(t) \\
V(t)/s
\end{bmatrix}
\begin{bmatrix}
\sigma_S \\
\sigma_{X_1}/s \\
\sigma_{X_2}/s
\end{bmatrix}
\frac{dB(t)}{X(t)} + \begin{bmatrix}
\sigma_{S} \\
\sigma_{X_1}/s \\
\sigma_{X_2}/s
\end{bmatrix}
\frac{dB(t)}{X(t)} + \begin{bmatrix}
m_0 \\
v_0/s
\end{bmatrix}.
\]

Collin-Dufresne et al. (2022) consider preferences with “source dependent” risk aversion, where the pricing of these shocks can be fully decoupled. Exponential models like ours correspond to one limiting case where all risk sources are priced in the same manner; in the LQ model of Garleanu and Pedersen (2013, 2016a) only price shocks are priced, whereas orthogonal state shocks do not enter the goal functional at all.

In the above, we have introduced an additional scaling parameter \(s\) to the setting of Koijen et al. (2009). The reason is that the value and momentum signals of Koijen et al. (2009) both fluctuate around zero, but with very different standard deviations making it difficult to compare their coefficients in the optimal trading strategies. We shall use the scaling parameter to equate the standard deviations of the value and momentum signals to facilitate comparison.
The corresponding price dynamics (converted from the geometric model of Koijen et al. (2009) into our arithmetic framework in line with Gärleanu and Pedersen (2013, 2016a)) then are

\[
dP(t) = \left( r (t) + \left( \frac{\mu_0 - r}{\mu} \right) + \left[ \phi (1 - \phi) \mu_1 \right] X(t) \right) dt + \left[ \sigma_S \right] \begin{pmatrix} 0 \\ \sigma_p \end{pmatrix} dB(t), \quad P(0) = p_0.
\]

### 4.2 Parameter Values

Koijen et al. (2009) estimate the following parameters using monthly data from 1946-2005 for an equally-weighted US stock index:

\[
\phi = 0.39, \quad \mu_0 = 0.0115, \quad \mu_1 = 0.012, \quad \alpha = 0.0094, \quad \sigma_S = 0.0628,
\]

\[
\sigma_{X_1} = -0.0843, \quad \sigma_{X_2} = 0.0152, \quad r = 0.0033,
\]

and use a standard coefficient of relative risk aversion \( \gamma \). We shall assume a fund size of $1bn, and in turn use a corresponding level of absolute risk aversion \( \gamma = r \beta = 5 \times 10^{-9} \). As a robustness check, we also explore the effect of smaller or larger levels of risk aversion below.

Collin-Dufresne et al. (2020) use proprietary execution data from the historical order databases of a large investment bank to estimate the trading costs on a universe of S&P500 stocks. They estimate quadratic transaction costs in four different volatility regimes, which vary from \( 1 \times 10^{-10} \) to \( 3 \times 10^{-10} \). We use the unconditional average\(^{11}\) of these estimates as our baseline specification and set \( \Lambda = 1.883 \times 10^{-10} \).\(^{12}\) The effect of smaller or higher trading costs is explored in another robustness check below.

To facilitate the comparisons between coefficients of the value and momentum signals later, we set the scaling parameter equal to \( s = \sqrt{0.390/0.00319} = 11.06 \). Then, the long-term variance of the re-scaled value factor equals the one of the momentum signal\(^{13}\). Finally, with a fund size of \( F = 10^9 \), the Hamiltonian in (3.10) can become poorly conditioned. An easy remedy is to notice that if the parameters \( \gamma \) and \( \Lambda \) both are scaled by fund size, \( \gamma \to \gamma F \) and \( \Lambda \to \Lambda F \) then the solutions to the Riccati equations are rescaled as follows: \( \Pi_{xx} \to \Pi_{xx}/F \), \( \Pi_{\theta\theta} \to \Pi_{\theta\theta}F \) and \( \Pi_x \to \Pi_x/F \), whereas \( \Pi_{\theta x} \) and \( \Pi_{\theta} \) remain unchanged.

### 4.3 Results for Baseline Model

For the baseline set of parameters described in the previous section, Table 1 compares the optimal policy from Theorem 3.1 to its frictionless counterpart from Koijen et al. (2009) and to the optimum.

---

\(^{11}\) More specifically, Collin-Dufresne et al. (2020) estimate a probability transition matrix of switching from one regime to another. From this we calculate the long-run probability of being in any regime, and take the probability-weighted average of the impact factors in the respective regimes.

\(^{12}\) This number is similar to the estimate in Engle et al. (2012) used by Gärleanu and Pedersen (2013) that a participation rate of 0.0159% has a market impact of 0.1%. Given an average dollar volume in S&P 500 stocks of $663mn (Frazzini et al. 2018), this equates to \( \Lambda = 0.95 \times 10^{-10} \). Frazzini et al. (2018) estimate a non-linear market impact model on a universe that includes some international stocks, but find a participation rate of 0.01% has an average impact factor of close to 0.1%, equating to another estimate of a similar magnitude: \( \Lambda = 1.51 \times 10^{-10} \).

\(^{13}\) To this end, we use the long-run covariance matrix \( X \) of the states process \( X(t) \) solves the Lyapunov equation

\[
A_x X + X A_x^T + \sigma_x \sigma_x^T = 0.
\]

For the parameters from the previous section, this leads to

\[
X = \begin{bmatrix} 0.00319 & -0.00393 \\ -0.00393 & 0.390 \end{bmatrix}.
\]

Our scaling factor in turn is the ratio of the two standard deviations.
mal portfolio with transaction costs and local mean-variance preferences derived by Găreanu and Pedersen (2013, 2016a).

<table>
<thead>
<tr>
<th>Portfolio Adjustments</th>
<th>Portfolio Constant</th>
<th>Trading Speed</th>
<th>Shadow Price Process</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Momentum (×10⁹)</td>
<td>Value (×10⁹)</td>
<td></td>
</tr>
<tr>
<td>Our Model</td>
<td>4.80</td>
<td>9.81</td>
<td>1.77</td>
</tr>
<tr>
<td>Kojien Model</td>
<td>16.39</td>
<td>9.82</td>
<td>1.85</td>
</tr>
<tr>
<td>GP Model</td>
<td>6.88</td>
<td>5.66</td>
<td>0.42</td>
</tr>
</tbody>
</table>

Table 1: Optimal portfolios for baseline parameters.

We see that relative to the frictionless model of Kojien et al. (2009), both the constant part of the portfolio and the adjustments to the relatively slow-moving value factor remain virtually unchanged in our model. The intuition for this is that transaction costs have little impact on the desired positions, as these positions are expected to be held for a considerable time due to the slow decay rate of the value signal. Both in Kojien et al. (2009) and in our model, the desired exposure to the value signal is significantly larger than in the model of Găreanu and Pedersen (2016a) due to intertemporal hedging, as innovations of the value factor are strongly negatively correlated with price shocks.

In contrast, the portfolio adjustments to the fast-moving momentum signal is reduced considerably in our model relative to the frictionless case. Now, the transaction cost of tightly tracking this signal is significant, and so our model scales back the desired position. In fact these desired positions become smaller than in the model of Garleanu and Pedersen (2016a). The reason is again that our model accounts for intertemporal hedging: innovations to the momentum factor are perfectly positively correlated with price shocks, and so to hedge changes to expected momentum returns, our optimal portfolio reduces exposure accordingly.

To understand how costs impact trading, we now look more closely at the shadow price process. Recall that the optimal target portfolio is the optimal portfolio for a frictionless problem where the price dynamics are described by the shadow price process from Theorem 3.2. At the end of Section 3.2, we discussed the two cost adjustments in the shadow price process. Columns 6 and 7 of Table 1 record the signal adjustments \( \hat{C}_x = C_x - \Pi_{\theta_x} A_x \) for the 3 portfolio strategies. In the frictionless model of Kojien et al. (2009) there are no costs and \( \hat{C}_x = C_x \). For both our model and the model of Garleanu and Pedersen (2016a) the impact of trading costs due to signal decay is similar. First, it significantly reduces the return to trading momentum, but slightly less, in our model. This reinforces the message above that the lower holding in momentum in our model is due in part to hedging motives. Second, and perhaps more surprisingly, this impact cost adjustment slightly increases the return to value. This is because a higher value signal reduces the decay to momentum offsetting some of the costs of trading out of momentum.

The other impact cost adjustment is to volatility. Columns 8 and 9 of Table 1 record \( \hat{\sigma}_p = \sigma_p - \Pi_{\theta_\sigma} \sigma_x \) for the 3 portfolio strategies. Neither the frictionless model of Kojien et al. (2009) nor the model of Garleanu and Pedersen (2016a) adjust volatility; the former as it is frictionless, the latter due to the certainty equivalence of their LQ objective. In our model, investors are risk averse.

---

14Here, the frictionless results correspond to the small transaction cost limit \( \Lambda \to 0 \) in Theorem 3.1. For the model of Garleanu and Pedersen (2016a) we use the same parameter values. The optimal trading rate of Garleanu and Pedersen (2016a) also depends on the time discount rate \( \delta \) (which is absorbed into the optimal consumption rate in our model). Here, we have set this parameter equal to the risk-free rate \( r \), but other choices such as \( \delta = 2r \) or \( \delta = r/2 \) produce very similar results.
to the additional trading costs resulting from stochastic changes in the investment opportunity set. It is the volatility of the momentum strategy that particularly increases the shadow volatility of the market due to these execution risks. This adjustment to volatility has two direct effects. First, in view of Equation (3.17), an increase in volatility reduces the optimal target positions. Secondly, this adjustment increases optimal trading speeds in our model by about 25% compared to the certainty equivalent setting of Garleanu and Pedersen (2016a); equation (3.22) implies that the optimal trading speed is roughly proportional to \( \frac{1}{\sqrt{\epsilon}} \). The intuition here is the trade-off in optimal execution emphasized in Almgren and Chriss (2001) between minimizing the impact cost of the trade and the risk that the price moves before the trade is executed. When costs are kept constant, higher volatility will imply higher trading speed due to the increased execution risks. The net impact of the volatility adjustment is, therefore, to reduce target positions but trade towards these positions more quickly.

Our model, therefore, includes the hedging motives emphasized in Koijen et al. (2009). It also includes the impact of costs on trading speeds discussed in Garleanu and Pedersen (2016a). But in addition to the adjustment to expected returns due to trading costs, our model includes an additional adjustment attributable to execution risks. Investors anticipate that the state signals will stochastically vary in the future, and because they are risk averse to these additional execution costs they scale back position and trade more quickly.

4.4 Robustness Checks

As a first robustness check, Table 2 varies the risk aversion parameter \( \gamma \) from the baseline calibration. We see that the portfolio constant and the adjustments due to the slow value factor are virtually proportional to risk tolerance \( \frac{1}{\gamma} \) like in the frictionless version of the model. In contrast, increasing risk aversion by a factor of ten only reduces the adjustment for the fast momentum factor by less than a factor of five. The reason is that when higher risk aversion reduces investment in this factor, then the haircut due to the high corresponding transaction costs is reduced, leading to a less than proportional adjustment. This observation is reflected in the expected return matrix, \( \hat{C}_x \), to the shadow price process; the returns to momentum increase with higher risk aversion in contrast to value which remains very nearly constant. The optimal trading speed increases roughly with the square root of risk aversion, in line with general small-cost asymptotics (Moreau et al., 2017).

| Risk Aversion \( \gamma \) | Portfolio Adjustments | Portfolio Trading Speed | Shadow Price Process
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( \times 10^9 )</td>
<td>( \times 10^9 )</td>
<td>( \times 10^9 )</td>
<td>( \hat{C}_x )</td>
</tr>
<tr>
<td>20</td>
<td>1.79</td>
<td>2.45</td>
<td>0.45</td>
</tr>
<tr>
<td>10</td>
<td>2.96</td>
<td>4.91</td>
<td>0.89</td>
</tr>
<tr>
<td>5</td>
<td>4.80</td>
<td>9.81</td>
<td>1.77</td>
</tr>
<tr>
<td>1</td>
<td>14.09</td>
<td>49.00</td>
<td>8.80</td>
</tr>
<tr>
<td>0.5</td>
<td>22.25</td>
<td>97.90</td>
<td>17.63</td>
</tr>
</tbody>
</table>

Table 2: Optimal portfolios with varying risk aversion.

Next, Table 3 varies the transaction cost parameter \( \Lambda \) from the baseline calibration. We see that the constant part of the portfolio and the persistent value factor remain virtually unaffected across a wide range of realistic cost parameters. In contrast, the sensitivity of the optimal portfolio to the fast momentum factor decreases considerably for higher trading costs. Again these are reflected

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in the shadow price return matrix $\widehat{C}_x$. The corresponding optimal trading speed roughly changes with the square root of the trading cost, again in line with general small-cost asymptotics (Moreau et al., 2017).

<table>
<thead>
<tr>
<th>Trading Cost Λ</th>
<th>Portfolio Adjustments</th>
<th>Portfolio Constant</th>
<th>Trading Speed</th>
<th>Shadow Price Process</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Momentum ($\times 10^9$)</td>
<td>Value ($\times 10^9$)</td>
<td>($\times 10^9$)</td>
<td>$\widehat{C}_x$</td>
</tr>
<tr>
<td>1.883</td>
<td>2.23</td>
<td>9.79</td>
<td>1.76</td>
<td>0.13</td>
</tr>
<tr>
<td>0.942</td>
<td>2.82</td>
<td>9.80</td>
<td>1.76</td>
<td>0.18</td>
</tr>
<tr>
<td>0.188</td>
<td>4.80</td>
<td>9.81</td>
<td>1.77</td>
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</tr>
<tr>
<td>0.038</td>
<td>7.55</td>
<td>9.82</td>
<td>1.79</td>
<td>0.87</td>
</tr>
<tr>
<td>0.019</td>
<td>8.86</td>
<td>9.82</td>
<td>1.80</td>
<td>1.20</td>
</tr>
<tr>
<td>0</td>
<td>16.39</td>
<td>9.82</td>
<td>1.85</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

Table 3: Optimal portfolios with varying transaction costs.

These comparative statics for the risk aversion and trading cost parameters parallel the ones obtained by Garleanu and Pedersen (2013, 2016a) for their mean-variance model. In contrast, varying the interest rate leads to very different conclusions, illustrated in Table 4. To wit, in Garleanu and Pedersen (2013, 2016a), the interest rate only affects the optimal target portfolio through the discount rate and in turn only causes minor changes across a broad range of parameters. In the exponential model, interest rates also have virtually no effect on how the short-lived momentum signal is used. However, high interest rates make it less attractive to realize gains from the value signal over longer horizons. As the interest rate increases, the value factor is in turn scaled back considerably whereas the role of the momentum factor remains almost unchanged. In summary, the interest rate therefore affects the target portfolio in virtually the opposite way as trading costs. The corresponding trading speed is largely insensitive to variations of the interest rate, though.

<table>
<thead>
<tr>
<th>Interest Rate</th>
<th>Portfolio Adjustments</th>
<th>Portfolio Constant</th>
<th>Trading Speed</th>
<th>Shadow Price Process</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Momentum ($\times 10^9$)</td>
<td>Value ($\times 10^9$)</td>
<td>($\times 10^9$)</td>
<td>$\widehat{C}_x$</td>
</tr>
<tr>
<td>0.01</td>
<td>4.60</td>
<td>5.14</td>
<td>0.09</td>
<td>0.42</td>
</tr>
<tr>
<td>0.005</td>
<td>4.70</td>
<td>7.66</td>
<td>0.86</td>
<td>0.42</td>
</tr>
<tr>
<td>0.0033</td>
<td>4.80</td>
<td>9.81</td>
<td>1.77</td>
<td>0.41</td>
</tr>
<tr>
<td>0.002</td>
<td>4.93</td>
<td>12.61</td>
<td>3.33</td>
<td>0.41</td>
</tr>
<tr>
<td>0.001</td>
<td>5.07</td>
<td>15.81</td>
<td>5.67</td>
<td>0.40</td>
</tr>
</tbody>
</table>

Table 4: Models with varying interest rates.

Finally, Table 5 explores the role of the correlation $\rho = \sigma_{X_1}/\sqrt{\sigma_{X_1}^2 + \sigma_{X_2}^2}$ between price and value shocks. The point estimate $\rho = -0.984$ of Koijen et al. (2009) implies a high negative correlation, which in turn leads to a strong intertemporal hedging motive. Table 5 illustrates the sensitivity of the optimal portfolio with respect to this parameter. We see that as the correlation falls, the market is a less valuable hedge against a fall in expected returns to value and so the portfolio adjustments to value reduce substantially. For $\rho = -0.75$ the portfolio adjustments for momentum and value are roughly equal, as in the baseline case for the Garleanu and Pedersen (2013) model.

Barberis (2000) also estimates a very strong negative correlation $\rho = -0.935$ between shocks to prices and dividend yields. This motivates, e.g., Wachter (2002) to study a model with perfect negative correlation $\rho = -1$. 15
model. However, accounting for the execution risk, \( \sigma_p \), means that portfolio adjustments are smaller and trading speeds are faster in our model than in theirs.

<table>
<thead>
<tr>
<th>Correlation</th>
<th>Portfolio Adjustments</th>
<th>Trading Speed</th>
<th>Shadow Price Process</th>
</tr>
</thead>
<tbody>
<tr>
<td>Momentum</td>
<td>Value</td>
<td>( C_x )</td>
<td>( \sigma_p )</td>
</tr>
<tr>
<td>(-0.9841)</td>
<td>( 4.80 \times 10^9 )</td>
<td>1.77</td>
<td>0.163 0.104</td>
</tr>
<tr>
<td>(-0.85)</td>
<td>( 4.54 \times 10^9 )</td>
<td>0.55</td>
<td>0.168 0.107</td>
</tr>
<tr>
<td>(-0.75)</td>
<td>( 4.48 \times 10^9 )</td>
<td>0.40</td>
<td>0.169 0.107</td>
</tr>
<tr>
<td>(-0.5)</td>
<td>( 4.41 \times 10^9 )</td>
<td>0.27</td>
<td>0.171 0.108</td>
</tr>
<tr>
<td>0</td>
<td>( 4.33 \times 10^9 )</td>
<td>0.19</td>
<td>0.172 0.108</td>
</tr>
</tbody>
</table>

Table 5: Models with varying correlation between price and value shocks.

5 Conclusion

This paper derives the optimal portfolio/consumption policy of a CARA investor who trades multiple risky assets with partially predictable returns in the presence of quadratic transaction costs. The optimal trading rule tracks a target portfolio that accounts for intertemporal hedging against both changing investment opportunities and future transaction costs. Likewise, the corresponding optimal trading speed takes into account both the short run risk of immediate price shocks and the long run risk modulated by changing investment opportunities. Our model thereby allows to study the interplay of some of the central features of frictionless dynamic portfolio theory and models with transaction costs.

A Control Heuristics

In this appendix, we sketch how to identify candidates for the value function and optimal controls for the lifetime portfolio/consumption problem \((2.4)\) using formal stochastic control arguments. A rigorous verification theorem is in turn presented in Appendix D.

The value function \( J(x, w, \theta) \) for \((2.4)\) depends on investment opportunities (through the mean-reverting component \( x \) of the expected returns), wealth \( w \), and on the positions \( \theta \) in the risky assets. Standard arguments suggest that it solves the HJB equation

\[
0 = -\delta J + \theta^\top \sigma_p \sigma_x^\top \frac{\partial^2 J}{\partial w \partial x} + \frac{1}{2} \text{tr} \left( \sigma_x \sigma_x^\top \frac{\partial^2 J}{\partial x^2} \right) + x^\top A_x^\top \frac{\partial J}{\partial x} + \frac{1}{2} \theta^\top \sigma_p \sigma_p^\top \theta \frac{\partial^2 J}{\partial w^2} + \sup_{c, \dot{\theta}} \left\{ \dot{\theta}^\top \frac{\partial J}{\partial \theta} + \left( rw - c + \theta^\top (\bar{\mu} + C_x x) - \frac{1}{2} \theta^\top \Lambda \theta \right) \frac{\partial J}{\partial w} - e^{-\beta c} \right\}. \tag{A.1}
\]

Pointwise optimization in turn yields the optimal controls in feedback form:

\[
c = -\frac{1}{\beta} \left( \ln \frac{\partial J}{\partial w} - \ln \beta \right), \quad \dot{\theta} = \Lambda^{-1} \frac{\partial J/\partial \theta}{\partial J/\partial w}. \tag{A.2}
\]

To proceed, we set \( \gamma = r\beta \) (this is the risk aversion of the indirect utility function \( J \)), and make the exponential-quadratic ansatz \((3.7)\) with coefficients \( \Pi_x x \in \mathbb{R}^{d \times d} \), \( \Pi_x \theta \in \mathbb{R}^{m \times d} \), \( \Pi_{\theta \theta} \in \mathbb{R}^{m \times m} \), \( \Pi_x \in \mathbb{R}^d \), \( \Pi_\theta \in \mathbb{R}^m \), \( J_0 \in \mathbb{R} \) to be determined. Then, the candidate optimal controls simplify
to \((3.8)\) and \((3.9)\). After plugging these expressions back into the HJB equation and cancelling the common term \(J\), it follows that

\[
0 = r - \delta - r \ln r J_0 - \frac{\gamma}{2} \text{tr} \left( \sigma_x^\top \Pi_{xx} \sigma_x \right) - \frac{\gamma}{2} (\Pi_{\theta\theta} + \Pi_{\theta} + \Pi_{\theta x} x)^\top \Lambda^{-1} (\Pi_{\theta\theta} + \Pi_{\theta} + \Pi_{\theta x} x)
\]

\[
- \gamma \theta^\top (\mu + C x) - \gamma x^\top A_x^\top (\Pi_{xx} x - \Pi_x - \Pi_{\theta x}^\top \theta) + \frac{1}{2} \gamma^2 \theta^\top \sigma_p \sigma_p^\top \theta 
\]

\[
+ \frac{\gamma^2}{2} \sigma_p \sigma_p^\top \left( \Pi_{xx} x - \Pi_x - \Pi_{\theta x}^\top \theta \right)^\top - r \gamma \left( -\frac{1}{2} x^\top \Pi_{xx} x + \Pi_x^\top x + \frac{1}{2} \theta^\top \Pi_{\theta\theta} \theta + \Pi_{\theta}^\top \theta + \theta^\top \Pi_{\theta x} x \right) 
\]

Comparison of coefficients for the terms independent of all state variables gives \((3.6)\). For the terms quadratic in \(x\) we obtain \((3.2)\). Finally, comparison of coefficients for the cross terms that depend both on \(\theta\) and \(x\) yields \((3.3)\).

Once an appropriate solution \((\Pi_{\theta\theta}, \Pi_{xx}, \Pi_{\theta x})\) of the coupled matrix Riccati equations \((3.1), (3.2), (3.3)\) is identified, the remaining coefficients of the candidate value function can be computed directly. To wit, comparison of coefficients for the terms that only depend on \(\theta\) and \(x\), respectively, leads to the linear equations \((3.4)\) and \((3.5)\) for \(\Pi_{\theta}\) and \(\Pi_{xx}\). Finally, comparison of coefficients for the terms independent of all state variables gives \((3.6)\).

### B Proof of Theorem 3.1(i)

Next, we establish global existence for the system of Riccati equations that defines our candidate value function \((3.7)\) and in turn the corresponding feedback controls \((3.8), (3.9)\).

#### B.1 Notation

We first introduce some notation and concepts that are used throughout the proof without further mention. The Kronecker product of matrices \(A \in \mathbb{R}^{m \times n}\) and \(B \in \mathbb{R}^{m' \times n'}\) is denoted by

\[
A \otimes B := \begin{bmatrix}
A_{11} B & \cdots & A_{1n} B \\
\vdots & \ddots & \vdots \\
A_{m1} B & \cdots & A_{mn} B
\end{bmatrix} \in \mathbb{R}^{mn \times mn'}.
\]

The vectorization \(\text{vec}(A)\) of a matrix \(A \in \mathbb{R}^{m \times n}\) is the \(mn \times 1\) column vector obtained by stacking the columns of the matrix \(A\) on top of one another:

\[
\text{vec}(A) = [a_{1,1}, \ldots, a_{m,1}, a_{1,2}, \ldots, a_{m,2}, \ldots, a_{1,n}, \ldots, a_{m,n}]^\top.
\]

The operator norm of a matrix or vector is denoted by \(\| \cdot \|\), whereas \(\| \cdot \|\) refers to the determinant of a square matrix. A symmetric matrix \(A \in \mathbb{R}^{m \times m}\) is positive semidefinite if \(z^\top A z \geq 0\) for all \(z \in \mathbb{R}^m\) and positive definite if this inequality is strict for all \(z \in \mathbb{R}^m \setminus \{0\}\). A matrix \(A \in \mathbb{R}^{m \times m}\) is stable if all the eigenvalues of \(A\) have strictly negative real parts.

#### B.2 Aggregation of the Riccati Equations

We start the proof of Theorem 3.1 by aggregating the three coupled matrix Riccati equations \((3.1), (3.2), (3.3)\) into a higher-dimensional system. To this end, write

\[
\Pi = \begin{bmatrix}
\Pi_{xx} & -\Pi_{\theta x}^\top \\
-\Pi_{\theta x} & -\Pi_{\theta \theta}
\end{bmatrix} \quad \text{(B.1)}
\]
and define the function $f : \mathbb{R}^{(d+m)\times(d+m)} \times \mathbb{R} \to \mathbb{R}^{(d+m)\times(d+m)}$ by

$$f(\Pi; \epsilon) = \gamma \Pi B_1 \sigma_x \sigma_x^T B_1^\top \Pi - \epsilon \Pi B_2 \Lambda^{-1} B_2^\top \Pi + \Pi A + A^\top \Pi + Q,$$

where

$$B_1 = \begin{bmatrix} I_d \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ I_m \end{bmatrix}, \quad A = \begin{bmatrix} \frac{r}{2} I_d - A_x & \gamma \sigma_x \sigma_p^T \\ \frac{r}{2} I_m \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & -C_x^{-1} \\ -C_x \sigma_p \sigma_p^T \end{bmatrix}.$$

For $\epsilon = 1$, the off-diagonal terms of $f(\Pi; \epsilon)$ then coincide with the left-hand side of equation (3.3) and the diagonal terms match the left-hand sides of equations (3.1) and (3.2), respectively. Therefore, solving equations (3.1)-(3.3) is equivalent to solving $f(\Pi; 1) = 0$.

**B.3 Evolution as a Function of the Cost Size**

The scalar parameter $\epsilon$ is introduced in (B.2) to modulate the size of the transaction costs in equations (3.1)-(3.3):

$$0 = \epsilon \Pi \theta \Lambda - \Pi \theta - \gamma (\Pi \theta_x \sigma_x - \sigma_p) (\Pi \theta_x \sigma_x - \sigma_p)^\top,$$

$$0 = \gamma \Pi \sigma_x \sigma_x^\top \Pi \sigma_x + r \Pi \sigma_x - A_x^\top \Pi \sigma_x - \Pi \sigma_x A_x - \epsilon \Pi \sigma_x \Lambda^{-1} \Pi \theta_x,$$

$$0 = \gamma \sigma_p \sigma_x^\top \Pi \sigma_x - C_x - \left(\epsilon \Lambda^{-1} \Pi \theta + \frac{r}{2} I_m\right)^\top \Pi \theta_x - \Pi \sigma_p \left(\frac{r}{2} I_d - A_x + \gamma \sigma_x \sigma_x^\top \Pi \sigma_x\right).$$

As $\epsilon$ tends to zero the costs become prohibitively large and we obtain an explicit solution, that corresponds to the no-trade solution for the lifetime consumption problem (2.4). To wit, for $\epsilon = 0$, (B.3), (B.4), (B.5) simplify to

$$0 = r \Pi \theta - \gamma (\Pi \theta_x \sigma_x - \sigma_p) (\Pi \theta_x \sigma_x - \sigma_p)^\top,$$

$$0 = \gamma \Pi \sigma_x \sigma_x^\top \Pi \sigma_x + r \Pi \sigma_x - A_x^\top \Pi \sigma_x - \Pi \sigma_x A_x,$$

$$0 = \Pi \sigma_p \left(r I_d - A_x + \gamma \sigma_x \sigma_x^\top \Pi \sigma_x\right) + C_x - \gamma \sigma_p \sigma_x^\top \Pi \sigma_x.$$

(B.7) is evidently solved by $\Pi \sigma_x = 0$, and this is the unique positive semidefinite solution by (Zhou et al. 1996, Theorem 13.7). With $\Pi \sigma_x = 0$, the other two equations (B.6) and (B.8) in turn have the unique explicit solutions

$$\Pi \sigma_p = \frac{\gamma}{r} \left(\sigma_p + C_x (r I_d - A_x)^{-1} \sigma_x\right) \left(\sigma_p + C_x (r I_d - A_x)^{-1} \sigma_x\right)^\top.$$

Here, $\Pi \sigma_p$ is positive definite. In summary, the aggregate system $f(\Pi; 0) = 0$ has the solution

$$\Pi_0 = \begin{bmatrix} 0 & C_x (r I_d - A_x)^{-1} \\ C_x (r I_d - A_x)^{-1} & -\frac{\gamma}{r} \left(\sigma_p + C_x (r I_d - A_x)^{-1} \sigma_x\right) \left(\sigma_p + C_x (r I_d - A_x)^{-1} \sigma_x\right)^\top \end{bmatrix}. $$

Starting from this fully-explicit solution of $f(\Pi; \epsilon)$ for $\epsilon = 0$ (infinitely large transaction costs), our goal now is to show that solutions also exist for larger $\epsilon$, corresponding to finite levels of transaction costs. To this end, notice that if solutions

$$\Pi(\epsilon) = \begin{bmatrix} \Pi \sigma_x(\epsilon) & -\Pi \sigma_p(\epsilon)^\top \\ -\Pi \sigma_p(\epsilon) & -\Pi \sigma_p(\epsilon)^\top \end{bmatrix}$$
of \( f(\Pi; \epsilon) = 0 \) exist and depend on \( \epsilon \) in a smooth manner, then we can differentiate the equation with respect to \( \epsilon \) and obtain a matrix-valued initial value problem that describes the evolution of the solution starting from the explicit large-cost limit:

\[
\Pi'(\epsilon) A(\Pi(\epsilon); \epsilon) + A(\Pi(\epsilon); \epsilon) \Pi'(\epsilon) - \Pi(\epsilon) B_2 \Lambda^{-1} B_2^\top \Pi(\epsilon) = 0, \quad \Pi(0) = \Pi_0, \tag{B.10}
\]

where

\[
A(\Pi; \epsilon) = \gamma B_1 \sigma_x \sigma_x^\top B_1^\top \Pi - \epsilon B_2 \Lambda^{-1} B_2^\top \Pi + A \tag{B.11}
\]

\[
= \left[ \begin{array}{ccc}
\gamma \sigma_x \sigma_x^\top \Pi_{xx} + \frac{\epsilon}{2} I_d - A_x & \gamma \sigma_x (\sigma_p - \Pi_{\theta x} \sigma_x)^\top \\
\epsilon \Lambda^{-1} \Pi_{\theta x} & \epsilon \Lambda^{-1} \Pi_{\theta \theta} + \frac{\epsilon}{2} I_m
\end{array} \right].
\]

### B.4 Wellposedness of the Evolution Equation

Using the properties of the matrix-valued function \( A(\cdot; \epsilon) \) established in Appendix B.5, we now prove the global existence and uniqueness of the matrix ODE (B.10) for arbitrary values of \( \epsilon \), without any constraints on the model parameters. In particular, this establishes the existence result for our original Riccati system (B.3), (B.4), (B.5), with \( \epsilon = 1 \) stated in Theorem B.1.

**Theorem B.1.**

(i) There exists a unique \( C^1 \) solution \( \epsilon \mapsto \Pi(\epsilon) \) of (B.10) on \([0, \infty)\).

(ii) For every \( v \in \mathbb{R}^{(d+m)\times 1} \), \( \epsilon \mapsto v^\top \Pi(\epsilon) v \) is increasing on \([0, \infty)\).

(iii) For every \( \epsilon \geq 0 \), \( f(\Pi(\epsilon); \epsilon) = 0 \), and \( B_1^\top \Pi(\epsilon) B_1 \in \mathbb{R}^{d \times d} \) and \( -B_2^\top \Pi(\epsilon) B_2 \in \mathbb{R}^{m \times m} \) are both symmetric positive-semidefinite matrices. Moreover, \( B_1^\top \Pi(\epsilon) B_1 \), \( -B_1^\top \Pi(\epsilon) B_2 \) and \( B_2^\top \Pi(\epsilon) B_2 \) solve the matrix Riccati equations (B.3), (B.4), (B.5).

**Proof.** We prove the assertions of the theorem as follows. First, we establish local existence for (B.10) on a maximum interval of existence \([0, \epsilon_+)\). Then, we show that properties (ii) and (iii) hold on \([0, \epsilon_+)\). Finally, we prove that the local solutions \( \Pi(\epsilon) \) remains uniformly bounded in \( \epsilon \). This implies that \( \epsilon_+ = \infty \), so that the solutions are in fact global and properties (ii) and (iii) in turn hold for arbitrary values of \( \epsilon \).

**Step 1: Establish local existence and uniqueness for (B.10).** Using the Kronecker product and the vectorization operator, the matrix ODE (B.10) can be rewritten as

\[
\begin{cases}
\text{vec}(\Pi(\epsilon))' = \left( I_{d+m} \otimes A(\Pi(\epsilon); \epsilon)^\top + A(\Pi(\epsilon); \epsilon)^\top \otimes I_{d+m} \right)^{-1} \text{vec} \left( \Pi(\epsilon) B_2 \Lambda^{-1} B_2^\top \Pi(\epsilon) \right), \\
\text{vec}(\Pi(0)) = \text{vec}(\Pi_0).
\end{cases} \tag{B.12}
\]

Here \((I_{d+m} \otimes A(\Pi; \epsilon)^\top + A(\Pi; \epsilon)^\top \otimes I_{d+m})\) is invertible if and only if the determinant of \( A \) is not 0. At \( \epsilon = 0 \),

\[
|A(\Pi(0); 0)| = \left| \begin{array}{ccc}
\frac{\epsilon}{2} I_d - A_x & \gamma \sigma_x (\sigma_p - \Pi_{\theta x}(0) \sigma_x)^\top \\
0 & \frac{\epsilon}{2} I_m
\end{array} \right| = \frac{\epsilon}{2} \left| \frac{\epsilon}{2} I_d - A_x \right| > 0.
\]

Hence there exists a neighborhood of \((\Pi(0), 0)\) such that \( A(\Pi; \epsilon) \) is invertible. By Cramer’s rule, each entry of the matrix \((I_{d+m} \otimes A(\Pi; \epsilon)^\top + A(\Pi; \epsilon)^\top \otimes I_{d+m})^{-1} \text{vec}(\Pi B_2 \Lambda^{-1} B_2^\top \Pi)\) is the ratio of determinants of two matrices, and therefore the ratio of two polynomials. Hence, the entries are jointly continuous in \((\Pi, \epsilon)\) and locally Lipschitz in \( \Pi \) in a neighborhood of \((\Pi(0), 0)\).
Step 2: Link to the original Riccati equations \((B.3) - (B.5)\) on \(\theta \in [0, \epsilon_+).\) As \(f(\Pi(0); 0) = 0,\) integrating both sides of the ODE \((B.10)\) with respect to \(\epsilon\) yields
\[
f(\Pi(\epsilon); \epsilon) = f(\Pi(0); 0) - f(\Pi(0); 0) = 0
\]

The solution of the matrix ODE \((B.10)\) therefore indeed yields a solution of \(f(\Pi(\epsilon); \epsilon) = 0,\) and in turn a solution of the coupled system of algebraic Riccati equations \((B.3) - (B.5)\) as asserted in (iii).

Step 3: Establish properties of the local solution. Next, we show that the matrices \(\Pi_{xx}(\epsilon)\) and \(\Pi_{\theta\theta}(\epsilon)\) are positive semidefinite for \(\epsilon \in [0, \epsilon_+).\) We start with \(\Pi_{\theta\theta}(\epsilon) = -B_1^T \Pi_1(\epsilon) B_2.\) Fixing \(\epsilon \in [0, \epsilon_+\) and treating \(\Pi_{\theta\theta}(\epsilon) = -B_1^T \Pi_1(\epsilon) B_2\) as given, \(\Pi_{\theta\theta}(\epsilon)\) is the solution \((B.3)\). By (Zhou et al. 1996 Theorem 13.7), positive semidefiniteness of \(\Pi_{\theta\theta}(\epsilon)\) is equivalent to stability of \(-\epsilon \Lambda^{-1/2} \Pi_{\theta\theta}(\epsilon) - \frac{r}{2} I_m.\) This is in turn equivalent to establishing that the eigenvalues of \(\epsilon \Lambda^{-1/2} \Pi_{\theta\theta}(\epsilon) \Lambda^{-1/2}\) are all greater than \(-r/2\) for every \(\epsilon \in [0, \epsilon_+).\) To this end, we assume by contradiction that there exists \(\epsilon_0 \in [0, \epsilon_+)\) such that the smallest eigenvalue of \(\epsilon_0 \Lambda^{-1/2} \Pi_{\theta\theta}(\epsilon_0) \Lambda^{-1/2}\) is less than or equal to \(-r/2.\) Define
\[
f_\theta(\epsilon) = \min \left\{ \epsilon \theta^T \Lambda^{-1/2} \Pi_{\theta\theta}(\epsilon) \Lambda^{-1/2} \theta : \theta \in \mathbb{R}^m, \| \theta \| = 1 \right\},
\]
which is continuous in \(\epsilon\) by the maximum theorem. As \(f_\theta(0) = 0\) and by the variational characterization of the smallest eigenvalue, it follows that there exists \(\epsilon_\theta \in (0, \epsilon_0]\) such that \(f_\theta(\epsilon_\theta) = -r/2.\) Again by the variational characterization of the smallest eigenvalue \(-r/2,\) there exists \(\theta(\epsilon_\theta) \in \mathbb{R}^m\) with \(\| \theta(\epsilon_\theta) \| = 1\) such that
\[
\epsilon_\theta \Lambda^{-1/2} \Pi_{\theta\theta}(\epsilon_\theta) \Lambda^{-1/2} \theta(\epsilon_\theta) = -\frac{r}{2} \theta(\epsilon_\theta).
\]
Since \(\epsilon_\theta > 0,\) it follows that
\[
\theta(\epsilon_\theta)^T \Lambda^{-1/2} \Pi_{\theta\theta}(\epsilon_\theta) \Lambda^{-1/2} \theta(\epsilon_\theta) = -\frac{1}{\epsilon_\theta} \frac{r}{2} \theta(\epsilon_\theta)^T \theta(\epsilon_\theta) = -\frac{r}{2 \epsilon_\theta}.
\]
After multiplying the Riccati equation \((B.3)\) by \((\Lambda^{-1/2} \theta(\epsilon_\theta))^T\) and \(\Lambda^{-1/2} \theta(\epsilon_\theta)\) from the left and right, respectively, and plugging in these two identities, we arrive at the desired contradiction:
\[
0 \leq \gamma \left( \Lambda^{-1/2} \theta(\epsilon_\theta) \right)^T \left( \Pi_{\theta\theta}(\epsilon_\theta) \sigma_x - \sigma_p \right) \left( \Pi_{\theta\theta}(\epsilon_\theta) \sigma_x - \sigma_p \right)^T \Lambda^{-1/2} \theta(\epsilon_\theta)
= \epsilon_\theta \left( \Lambda^{-1/2} \theta(\epsilon_\theta) \right)^T \Pi_{\theta\theta}(\epsilon_\theta) \Lambda^{-1/2} \theta(\epsilon_\theta) + r \left( \Lambda^{-1/2} \theta(\epsilon_\theta) \right)^T \Pi_{\theta\theta}(\epsilon_\theta) \Lambda^{-1/2} \theta(\epsilon_\theta)
= \frac{1}{\epsilon_\theta} \left\| \epsilon_\theta \Lambda^{-1/2} \Pi_{\theta\theta}(\epsilon_\theta) \Lambda^{-1/2} \theta(\epsilon_\theta) \right\|^2 + r \theta(\epsilon_\theta)^T \Lambda^{-1/2} \Pi_{\theta\theta}(\epsilon_\theta) \Lambda^{-1/2} \theta(\epsilon_\theta)
= \frac{r^2}{4 \epsilon_\theta} - \frac{r^2}{2 \epsilon_\theta} = -\frac{r^2}{4 \epsilon_\theta} < 0.
\]
Therefore, for arbitrary \(\epsilon \in [0, \epsilon_+),\) we can conclude that \(-\epsilon \Lambda^{-1} \Pi_{\theta\theta}(\epsilon) - \frac{r}{2} I_m\) is stable. Hence, the positive semidefiniteness of \(\Pi_{\theta\theta}\) follows from (Zhou et al. 1996 Theorem 13.7).
To establish the positive semidefiniteness of \( \Pi_{xx}(\epsilon) = B_1^\top \Pi(\epsilon) B_1 \), we first show the positive definiteness of \( \Pi_{xx}(\epsilon) + \tilde{A}_x \), where \( \tilde{A}_x \) is the unique positive definite solution of

\[
\gamma \tilde{A}_x \sigma_x \sigma_x^\top \tilde{A}_x + A_x^\top \tilde{A}_x + \tilde{A}_x A_x = 0.
\]

(Existence and uniqueness once again follow from \( \text{(Zhou et al., 1996, Theorem 13.7)} \).) Suppose by contradiction that for the smallest eigenvalue of \( \Pi_{xx}(\epsilon) + \tilde{A}_x \) is less than or equal to 0 for some \( \epsilon \in [0, \epsilon_+) \). As above, the continuity of the smallest eigenvalues and the existence of the corresponding eigenvectors show that there exists \( \epsilon_x \in [0, \epsilon_+) \) and \( x(\epsilon_x) \in \mathbb{R}^d \) with \( \|x(\epsilon_x)\| = 1 \) such that

\[
\Pi_{xx}(\epsilon_x)x(\epsilon_x) = -\tilde{A}_x x(\epsilon_x).
\]

Multiplying the Riccati equation (B.4) by \( x(\epsilon_x)^\top \) and \( x(\epsilon_x) \) from the left and right, respectively, plaguing in this identity and rearranging terms, we obtain

\[
0 \leq \epsilon_x x(\epsilon_x)^\top \Pi \theta_x (\epsilon_x)^\top \Pi \theta_x (\epsilon_x) x(\epsilon_x)
= x(\epsilon_x)^\top \left( \gamma \Pi_{xx}(\epsilon_x) \sigma_x \sigma_x^\top \Pi_{xx}(\epsilon_x) + r \Pi_{xx}(\epsilon_x) - A_x^\top \Pi_{xx}(\epsilon_x) - \Pi_{xx}(\epsilon_x) A_x \right) x(\epsilon_x)
= \gamma x(\epsilon_x)^\top \tilde{A}_x \sigma_x \sigma_x^\top \tilde{A}_x x(\epsilon_x) + x(\epsilon_x)^\top A_x^\top \tilde{A}_x x(\epsilon_x) + x(\epsilon_x)^\top A_x \tilde{A}_x x(\epsilon_x)
= -r x(\epsilon_x)^\top \tilde{A}_x x(\epsilon_x) + x(\epsilon_x)^\top \left( \gamma \tilde{A}_x \sigma_x \sigma_x^\top \tilde{A}_x + A_x^\top \tilde{A}_x + \tilde{A}_x A_x \right) x(\epsilon_x)
= -r x(\epsilon_x)^\top \tilde{A}_x x(\epsilon_x) < 0.
\]

This contradicts the positive definiteness of \( \tilde{A}_x \), so \( \Pi_{xx}(\epsilon) + \tilde{A}_x \) is indeed positive definite as asserted above. Next, notice that

\[
\left( \Pi_{xx}(\epsilon) + \tilde{A}_x \right) \left( \gamma \sigma_x \sigma_x^\top \Pi_{xx}(\epsilon) + \frac{r}{2} I_d - A_x \right) + \left( \gamma \sigma_x \sigma_x^\top \Pi_{xx}(\epsilon) + \frac{r}{2} I_d - A_x \right)^\top \left( \Pi_{xx}(\epsilon) + \tilde{A}_x \right)
= r \tilde{A}_x + \epsilon \Pi \theta_x (\epsilon)^\top A_x \sigma_x \sigma_x^\top \Pi_{xx}(\epsilon) + \Pi_{xx}(\epsilon) \sigma_x \sigma_x^\top \Pi_{xx}(\epsilon) \sigma_x \sigma_x + A_x \sigma_x \sigma_x^\top \Pi_{xx}(\epsilon)
- A_x^\top \tilde{A}_x - \tilde{A}_x A_x
= r \tilde{A}_x + \epsilon \Pi \theta_x (\epsilon)^\top A_x \sigma_x \sigma_x^\top \Pi_{xx}(\epsilon) + \gamma \Pi_{xx}(\epsilon) \sigma_x \sigma_x^\top \Pi_{xx}(\epsilon) \sigma_x \sigma_x + A_x \sigma_x \sigma_x^\top \Pi_{xx}(\epsilon)
+ \gamma A_x \sigma_x \sigma_x^\top \Pi_{xx}(\epsilon)
= r \tilde{A}_x + \epsilon \Pi \theta_x (\epsilon)^\top A_x \sigma_x \sigma_x^\top \Pi_{xx}(\epsilon) + \gamma \left( \Pi_{xx}(\epsilon) + \tilde{A}_x \right) \sigma_x \sigma_x^\top \left( \Pi_{xx}(\epsilon) + \tilde{A}_x \right)
\]

is also positive definite since it is the sum of a positive definite matrix and two positive semidefinite matrices. As a consequence, the Lyapunov Lemma as in \( \text{(Zhou et al., 1996, Lemma 3.18)} \) yields the stability of \(- \left( \gamma \sigma_x \sigma_x^\top \Pi_{xx}(\epsilon) + \frac{r}{2} I_d - A_x \right) \). As the matrix \( \Pi_{xx}(\epsilon) \) solves the Riccati equation (B.4) (for fixed \( \epsilon \in [0, \epsilon_+) \) and \( \Pi \theta_x (\epsilon) \)), it in turn follows from \( \text{(Zhou et al., 1996, Theorem 13.7)} \) that \( \Pi_{xx}(\epsilon) \) is indeed positive semidefinite for all \( \epsilon \in [0, \epsilon_+) \).

**Step 4:** Establish the monotonicity of the local solution. From Steps 2 and 3, we know that \(-B_2^\top \Pi(\epsilon) B_2 \) and \(B_1^\top \Pi(\epsilon) B_1 \) are the positive semidefinite solutions of the matrix Riccati equations (B.3) and (B.4) respectively. Hence, by Lemma B.3, the matrix \(-A(\Pi(\epsilon); \epsilon)\) from (B.11) is stable. As a consequence\(^{16}\), we can therefore rewrite (B.10) as

\[
\Pi'(\epsilon) = \int_0^\infty e^{-A(\Pi(\epsilon); \epsilon)^\top \Pi(\epsilon) B_2 A^{-1} B_2^\top \Pi(\epsilon) e^{-A(\Pi(\epsilon); \epsilon)^\top} d\tau.
\]

\(^{16}\)For a strictly negative number \(a < 0\) any real \(b\) can be written as \(b = \int_0^\infty e^{\tau a} ab \, d\tau\). Likewise, for a stable matrix \(A \in \mathbb{R}^m\), we can also represent an arbitrary matrix \(B \in \mathbb{R}^m\) as \(B = \int_0^\tau e^{A^\top \tau} (A^\top B + BA) e^{A \tau} \, d\tau\).
This shows that $\Pi'(\epsilon)$ is positive semidefinite for every $\epsilon \in [0, \epsilon_+]$ by Zhou et al. 1996 Lemma 3.18, and in turn yields the asserted monotonicity:

$$\left(v^\top \Pi(\epsilon) v\right)' = v^\top \Pi'(\epsilon) v \geq 0, \quad \text{for } v \in \mathbb{R}^{d+m}.$$  

**Step 5: Show that the local solution is global.** To ease notation, define

$$y(\epsilon) := \Lambda^{-1/2} (\Pi_{\theta x}(\epsilon) \sigma_x - \sigma_p) \in \mathbb{R}^{m \times k},$$  

and notice that $\gamma \epsilon y(\epsilon) y(\epsilon)^\top + \frac{r^2}{4} I_m$ is positive definite for every $\epsilon \in (0, \epsilon_+)$. With this notation, (B.3) can be rewritten as

$$\Lambda^{-1/2} \Pi_{\theta x}(\epsilon) = - \left( \gamma \epsilon y(\epsilon) y(\epsilon)^\top + \frac{r^2}{4} I_m \right)^{-1/2} \times \left( \gamma y(\epsilon) \sigma_x^\top \Pi_{xx}(\epsilon) + \Lambda^{-1/2} \left( \Pi_{\theta x}(\epsilon) \left( \frac{r}{2} I_d - A_x \right) + C_x \right) \right).$$  

Now, we prove by contradiction that the local solution is global. Suppose the maximum interval of existence is finite, $\epsilon_+ \in (0, \infty)$. Then, $\lim_{\epsilon \to \epsilon_+} \|\Pi(\epsilon)\| = \infty$ and in turn $\lim_{\epsilon \to \epsilon_+} \|\Pi_{\theta x}(\epsilon)\| = \infty$, as $\Pi_{\theta\theta}(\epsilon)$ and $\Pi_{xx}(\epsilon)$ would both remain bounded by (B.3) and (B.4) if $\Pi_{\theta x}(\epsilon)$ does not blow up.

To work towards a contradiction, we choose a sequence $\epsilon_n \to \epsilon_+$ such that $\|\Pi_{\theta x}(\epsilon_n)\| > 0$ and the following normalized limit exists:

$$\widehat{\Pi}_{\theta x} := \lim_{n \to \infty} \frac{\Lambda^{-1/2} \Pi_{\theta x}(\epsilon_n) (\sigma_x \sigma_x^\top)^{1/2}}{\|\Lambda^{-1/2} \Pi_{\theta x}(\epsilon_n) (\sigma_x \sigma_x^\top)^{1/2}\|}, \quad \text{with } \|\widehat{\Pi}_{\theta x}\| = 1.$$  

By the uniqueness of the positive semidefinite square root of positive semidefinite matrices, it in turn follows that

$$\lim_{n \to \infty} \frac{\left( (\sigma_x \sigma_x^\top)^{1/2} \Pi_{\theta x}(\epsilon_n) (\sigma_x \sigma_x^\top)^{1/2}\right)^{1/2}}{\|\Lambda^{-1/2} \Pi_{\theta x}(\epsilon_n) (\sigma_x \sigma_x^\top)^{1/2}\|^2} = \left( \widehat{\Pi}_{\theta x}^\top \widehat{\Pi}_{\theta x} \right)^{1/2}.$$  

Since 1 is the operator norm of $\widehat{\Pi}_{\theta x}$, it is also one of its singular values. Let $\theta \in \mathbb{R}^m$, $x \in \mathbb{R}^d$ with $\|\theta\| = 1 = \|x\|$ be a pair of singular vectors of $\widehat{\Pi}_{\theta x}$ with respect to the singular value 1, i.e.,

$$\widehat{\Pi}_{\theta x} x = \theta, \quad \widehat{\Pi}_{\theta x}^\top \theta = x. \quad (B.16)$$  

Then, $\widehat{\Pi}_{\theta x}^\top \widehat{\Pi}_{\theta x} x = \widehat{\Pi}_{\theta x}^\top \theta = x$ and it in turn follows from the singular value decomposition as in (Horn and Johnson 1991 Chapter 3) that

$$\left( \widehat{\Pi}_{\theta x}^\top \widehat{\Pi}_{\theta x} \right)^{1/2} x = x.$$  

Next, we show that $x$ is an eigenvector of $(\sigma_x \sigma_x^\top)^{1/2} \Pi_{xx}(\epsilon_n) (\sigma_x \sigma_x^\top)^{1/2}/\|\sigma_x \sigma_x^\top\|^{1/2} \Pi_{xx}(\epsilon_n) (\sigma_x \sigma_x^\top)^{1/2}/\|\sigma_x \sigma_x^\top\|^{1/2}$ as $n \to \infty$. Multiplying $(\sigma_x \sigma_x^\top)^{1/2}$ on both sides and rearranging terms, (B.4) can be written as

$$\gamma \epsilon_n (\sigma_x \sigma_x^\top)^{1/2} \Pi_{\theta x}(\epsilon_n) \Lambda^{-1} \Pi_{\theta x}(\epsilon_n) (\sigma_x \sigma_x^\top)^{1/2} = \left( \gamma (\sigma_x \sigma_x^\top)^{1/2} \Pi_{xx}(\epsilon_n) (\sigma_x \sigma_x^\top)^{1/2} + \frac{r}{2} I_d - (\sigma_x \sigma_x^\top)^{-1/2} A_x (\sigma_x \sigma_x^\top)^{1/2} \right)^\top$$  

$$\times \left( \gamma (\sigma_x \sigma_x^\top)^{1/2} \Pi_{xx}(\epsilon_n) (\sigma_x \sigma_x^\top)^{1/2} + \frac{r}{2} I_d - (\sigma_x \sigma_x^\top)^{-1/2} A_x (\sigma_x \sigma_x^\top)^{1/2} \right) - \left( \frac{r}{2} I_d - (\sigma_x \sigma_x^\top)^{-1/2} A_x (\sigma_x \sigma_x^\top)^{1/2} \right)^\top.$$  

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After normalizing by $\epsilon_n \| \Lambda^{-1/2} \Pi_{\theta x} (\epsilon_n)(\sigma_x \sigma_x^T)^{1/2} \|^2$ and sending $n \to \infty$, the constant matrices terms vanish and we end up with:

$$
\lim_{n \to \infty} \frac{(\sigma_x \sigma_x^T)^{1/2} \Pi_{xx} (\epsilon_n)(\sigma_x \sigma_x^T)^{1/2}}{\epsilon_n \| \Lambda^{-1/2} \Pi_{\theta x} (\epsilon_n)(\sigma_x \sigma_x^T)^{1/2} \|^2} = \lim_{n \to \infty} \frac{(\sigma_x \sigma_x^T)^{1/2} \Pi_{xx} (\epsilon_n) \Lambda^{-1} \Pi_{\theta x} (\epsilon_n)(\sigma_x \sigma_x^T)^{1/2}}{\epsilon_n \| \Lambda^{-1/2} \Pi_{\theta x} (\epsilon_n)(\sigma_x \sigma_x^T)^{1/2} \|^2} = \hat{\Pi}_{\theta x}^T \hat{\Pi}_{\theta x}.
$$

After taking the square root and multiplying by the singular vector $x$ from the right, this leads to

$$
\lim_{n \to \infty} \left( (\sigma_x \sigma_x^T)^{1/2} \Pi_{xx} (\epsilon_n)(\sigma_x \sigma_x^T)^{1/2} x \right) = \left( \hat{\Pi}_{\theta x}^T \hat{\Pi}_{\theta x} \right)^{1/2} x = x. \tag{B.18}
$$

Now, we focus on $y(\epsilon_n)$ from (B.14). First notice that

$$
\lim_{n \to \infty} \left\| y(\epsilon_n) (\sigma_x \sigma_x^T)^{-1/2} \right\| = \lim_{n \to \infty} \left\| \Lambda^{-1/2} \Pi_{\theta x} (\epsilon_n)(\sigma_x \sigma_x^T)^{1/2} \Lambda^{-1} \Pi_{\theta x} (\epsilon_n)(\sigma_x \sigma_x^T)^{1/2} \right\| = \infty.
$$

Next, we show that $(\theta, (\sigma_x \sigma_x^T)^{-1/2} x)$ is a pair of singular vectors for $y(\epsilon_n)/\| y(\epsilon_n) \|$ as $n \to \infty$ with respect to the singular value 1. By definition,

$$
\lim_{n \to \infty} \left\| y(\epsilon_n) (\sigma_x \sigma_x^T)^{-1/2} \right\| = \lim_{n \to \infty} \left\| \Lambda^{-1/2} \Pi_{\theta x} (\epsilon_n)(\sigma_x \sigma_x^T)^{1/2} \right\| = 1,
$$

and therefore,

$$
\lim_{n \to \infty} \frac{\| y(\epsilon_n) \|}{\Lambda^{-1/2} \Pi_{\theta x} (\epsilon_n)(\sigma_x \sigma_x^T)^{1/2}} = \lim_{n \to \infty} \frac{\| y(\epsilon_n) \|}{\Lambda^{-1/2} \Pi_{\theta x} (\epsilon_n)(\sigma_x \sigma_x^T)^{1/2}} = 1. \tag{B.19}
$$

As a consequence, as $n \to \infty$,

$$
\lim_{n \to \infty} \frac{\theta^T y(\epsilon_n)}{\| y(\epsilon_n) \|} = \lim_{n \to \infty} \frac{\theta^T \Lambda^{-1/2} \Pi_{\theta x} (\epsilon_n)(\sigma_x \sigma_x^T)^{-1} \sigma_x}{\Lambda^{-1/2} \Pi_{\theta x} (\epsilon_n)(\sigma_x \sigma_x^T)^{1/2}} = \theta^T \hat{\Pi}_{\theta x} (\sigma_x \sigma_x^T)^{-1/2} \sigma_x = x^T (\sigma_x \sigma_x^T)^{-1/2} \sigma_x. \tag{B.20}
$$

Likewise,

$$
\lim_{n \to \infty} \frac{y(\epsilon_n) (\sigma_x \sigma_x^T)^{-1/2} x}{\| y(\epsilon_n) \|} = \lim_{n \to \infty} \frac{\Lambda^{-1/2} \Pi_{\theta x} (\epsilon_n)(\sigma_x \sigma_x^T)^{-1/2} x}{\Lambda^{-1/2} \Pi_{\theta x} (\epsilon_n)(\sigma_x \sigma_x^T)^{1/2}} = \hat{\Pi}_{\theta x} x = \theta. \tag{B.21}
$$

Therefore $(\theta, (\sigma_x \sigma_x^T)^{-1/2} x)$ is indeed a pair of singular vectors with respect to singular value 1, and

$$
\lim_{n \to \infty} \frac{y(\epsilon_n) y(\epsilon_n)^T \theta}{\| y(\epsilon_n) \|^2} = \lim_{n \to \infty} \frac{y(\epsilon_n) y(\epsilon_n)^T (\sigma_x \sigma_x^T)^{-1/2} x}{\| y(\epsilon_n) \|^2} = \theta.
$$

Thus,

$$
\lim_{n \to \infty} \frac{(y(\epsilon_n) y(\epsilon_n)^T)^{1/2} \theta}{\| y(\epsilon_n) \|} = \theta.
$$
Furthermore, $\theta$ is an eigenvector for $\|y(\epsilon_n)\| \left( \epsilon_n y(\epsilon_n) y(\epsilon_n)^T + \frac{\epsilon^2}{4} I_m \right)^{-1/2}$ as $n \to \infty$, because

$$\theta = \lim_{n \to \infty} \frac{\|y(\epsilon_n)\| \left( \epsilon_n y(\epsilon_n) y(\epsilon_n)^T + \frac{\epsilon^2}{4} I_m \right)^{-1/2} \left( \epsilon_n y(\epsilon_n) y(\epsilon_n)^T + \frac{\epsilon^2}{4} I_m \right)^{1/2}}{\|y(\epsilon_n)\|}$$

$$= \lim_{n \to \infty} \|y(\epsilon_n)\| \left( \epsilon_n y(\epsilon_n) y(\epsilon_n)^T + \frac{\epsilon^2}{4} I_m \right)^{-1/2} \left( \epsilon_n y(\epsilon_n) y(\epsilon_n)^T + \frac{\epsilon^2}{4} I_m \right)^{1/2} \theta$$

$$= \lim_{n \to \infty} \sqrt{\epsilon_n} \|y(\epsilon_n)\| \left( \epsilon_n y(\epsilon_n) y(\epsilon_n)^T + \frac{\epsilon^2}{4} I_m \right)^{-1/2} \theta$$

$$= \theta,$$

Finally, we show $(\theta, \sigma_x^T (\sigma_x \sigma_x^T)^{-1/2} x)$ is also a pair of singular vectors with respect to singular value $1$ for $\sqrt{\epsilon_n} \left( \epsilon_n y(\epsilon_n) y(\epsilon_n)^T + \frac{\epsilon^2}{4} I_m \right)^{-1/2} y(\epsilon_n)$ as $n \to \infty$. To this end, by (B.21) and (B.22), notice

$$\lim_{n \to \infty} \sqrt{\epsilon_n} \left( \epsilon_n y(\epsilon_n) y(\epsilon_n)^T + \frac{\epsilon^2}{4} I_m \right)^{-1/2} y(\epsilon_n) \sigma_x^T (\sigma_x \sigma_x^T)^{-1/2} x$$

$$= \lim_{n \to \infty} \sqrt{\epsilon_n} \|y(\epsilon_n)\| \left( \epsilon_n y(\epsilon_n) y(\epsilon_n)^T + \frac{\epsilon^2}{4} I_m \right)^{-1/2} \|y(\epsilon_n)\|^{-1} \frac{y(\epsilon_n) \sigma_x^T (\sigma_x \sigma_x^T)^{-1/2} x}{\|y(\epsilon_n)\|}$$

$$= \lim_{n \to \infty} \sqrt{\epsilon_n} \|y(\epsilon_n)\| \left( \epsilon_n y(\epsilon_n) y(\epsilon_n)^T + \frac{\epsilon^2}{4} I_m \right)^{-1/2} \theta$$

$$= \theta,$$

and using (B.22) then (B.20), we have

$$\lim_{n \to \infty} \theta^T \sqrt{\epsilon_n} \left( \epsilon_n y(\epsilon_n) y(\epsilon_n)^T + \frac{\epsilon^2}{4} I_m \right)^{-1/2} y(\epsilon_n) \theta$$

$$= \lim_{n \to \infty} \sqrt{\epsilon_n} \|y(\epsilon_n)\| \|y(\epsilon_n)\|^{-1} \frac{y(\epsilon_n) \theta^T (\epsilon_n y(\epsilon_n) y(\epsilon_n)^T + \frac{\epsilon^2}{4} I_m)^{-1/2} \theta}{\|y(\epsilon_n)\|}$$

$$= \lim_{n \to \infty} \theta^T y(\epsilon_n) \frac{\theta}{\|y(\epsilon_n)\|}$$

$$= x^T (\sigma_x \sigma_x^T)^{-1/2} \sigma_x^T.$$

Then, $\theta^T \left( \epsilon_n y(\epsilon_n) y(\epsilon_n)^T + \frac{\epsilon^2}{4} I_m \right)^{-1/2}$ converges to $0$ as $n \to \infty$, and for arbitrary $n$, we have

$$\|A^{-1/2} \Pi_{\theta x}(\epsilon_n) \left( \frac{2}{\epsilon} I_d - A_x \right) (\sigma_x \sigma_x^T)^{1/2} \|$$

$$= \left\| A^{-1/2} \Pi_{\theta x}(\epsilon_n) (\sigma_x \sigma_x^T)^{1/2} \frac{2}{\epsilon} I_d - A_x \right\| \left( \sigma_x \sigma_x^T \right)^{1/2}$$

$$\leq \left\| A^{-1/2} \Pi_{\theta x}(\epsilon_n) (\sigma_x \sigma_x^T)^{1/2} \right\| \left\| \left( \sigma_x \sigma_x^T \right)^{1/2} \frac{2}{\epsilon} I_d - A_x \right\| \left( \sigma_x \sigma_x^T \right)^{1/2}$$

$$= \left\| \left( \sigma_x \sigma_x^T \right)^{-1/2} \frac{2}{\epsilon} I_d - A_x \right\| \left( \sigma_x \sigma_x^T \right)^{1/2} < \infty,$$

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and we can infer that

\[
\lim_{n \to \infty} \frac{\theta^T \left( e_n y(e_n) y(e_n)^T + \frac{r^2}{4} I_m \right)^{-1/2} \Lambda^{-1/2} \Pi_{\theta x}(e_n) \left( \frac{r}{2} I_d - A_x \right) (\sigma_x \sigma_x^T)^{1/2}}{\|\Lambda^{-1/2} \Pi_{\theta x}(e_n) (\sigma_x \sigma_x^T)^{1/2}\|} \\
\leq \lim_{n \to \infty} \frac{\theta^T \left( e_n y(e_n) y(e_n)^T + \frac{r^2}{4} I_m \right)^{-1/2} \gamma(e_n) \sigma_x^T \Pi_{xx}(e_n) + \Lambda^{-1/2} \Pi_{\theta x}(e_n) \left( \frac{r}{2} I_d - A_x \right) (\sigma_x \sigma_x^T)^{1/2}}{\|\Lambda^{-1/2} \Pi_{\theta x}(e_n) (\sigma_x \sigma_x^T)^{1/2}\|} \\
\leq \lim_{n \to \infty} \frac{\theta^T \left( e_n y(e_n) y(e_n)^T + \frac{r^2}{4} I_m \right)^{-1/2} \left( (\sigma_x \sigma_x^T)^{-1/2} \left( \frac{r}{2} I_d - A_x \right) (\sigma_x \sigma_x^T)^{1/2} \right)}{0} = 0. \quad (B.24)
\]

Finally, we multiply the right-hand side of (B.15) by \((\sigma_x \sigma_x^T)^{1/2}\) from the right and normalize by \(\|\Lambda^{-1/2} \Pi_{\theta x}(e_n) (\sigma_x \sigma_x^T)^{1/2}\|\). Then by (B.24) in the first step, (B.23) in the second and (B.18) in the third, we have

\[
- \lim_{n \to \infty} \frac{\theta^T \left( e_n y(e_n) y(e_n)^T + \frac{r^2}{4} I_m \right)^{-1/2} \gamma(e_n) \sigma_x^T \Pi_{xx}(e_n) + \Lambda^{-1/2} \Pi_{\theta x}(e_n) \left( \frac{r}{2} I_d - A_x \right) (\sigma_x \sigma_x^T)^{1/2}}{\|\Lambda^{-1/2} \Pi_{\theta x}(e_n) (\sigma_x \sigma_x^T)^{1/2}\|} \\
= - \lim_{n \to \infty} \frac{\theta^T \left( e_n y(e_n) y(e_n)^T + \frac{r^2}{4} I_m \right)^{-1/2} \gamma(e_n) \sigma_x^T \Pi_{xx}(e_n) (\sigma_x \sigma_x^T)^{1/2}}{\|\Lambda^{-1/2} \Pi_{\theta x}(e_n) (\sigma_x \sigma_x^T)^{1/2}\|} \\
= - \lim_{n \to \infty} \frac{x^T (\sigma_x \sigma_x^T)^{-1/2} \sigma_x \sigma_x^T \Pi_{xx}(e_n) (\sigma_x \sigma_x^T)^{1/2}}{\sqrt{\frac{\| \Lambda^{-1/2} \Pi_{\theta x}(e_n) (\sigma_x \sigma_x^T)^{1/2}\|}{\| \Lambda^{-1/2} \Pi_{\theta x}(e_n) (\sigma_x \sigma_x^T)^{1/2}\|}}} = -x^T.
\]

However, multiplying the left hand side of (B.15) by \((\sigma_x \sigma_x^T)^{1/2}\) from the right, normalizing by \(\|\Lambda^{-1/2} \Pi_{\theta x}(e_n) (\sigma_x \sigma_x^T)^{1/2}\|\) and using that \(\theta\) is a singular vector for \(\Pi_{\theta x}\) (cf. (B.16)), we obtain

\[
\lim_{n \to \infty} \frac{\theta^T \Lambda^{-1/2} \Pi_{\theta x}(e_n) (\sigma_x \sigma_x^T)^{1/2}}{\|\Lambda^{-1/2} \Pi_{\theta x}(e_n) (\sigma_x \sigma_x^T)^{1/2}\|} = \theta^T \Pi_{\theta x} = x^T.
\]

As the singular vector \(x\) has unit norm, this is the desired contradiction and we therefore have \(\epsilon_+ = \infty\).

**Remark B.2.** By replacing the sequence converging to \(\epsilon_+ \in (0, \infty)\) with a sequence that tends to \(\infty\), the argument from Step 5 of the proof of Theorem B.1 in fact shows that \(\Pi_{\theta x}(\epsilon)\) is not just uniformly bounded on compacts but even uniformly bounded on \([0, \infty)\).

### B.5 Properties of the Matrix Function \(A(\Pi; \epsilon)\)

The monotonicity of the solution \(\epsilon \mapsto \Pi(\epsilon)\) of the matrix ODE (B.10) (cf. Theorem B.1(iii)) is established using the stability of the matrices \(-A(\Pi; \epsilon)\) from (B.11). This property is established in the following auxiliary result:

**Lemma B.3.** For given \(\epsilon > 0\) and arbitrary \(\Pi_{\theta x} \in \mathbb{R}^{m \times d}\), let \(\Pi_{\theta \theta}\) and \(\Pi_{xx}\) be the unique stabilizing solutions of the Riccati equations\(^{17}\) (B.3) and (B.4). Then the corresponding matrix

\[
-A \left( \begin{bmatrix} \Pi_{xx} - \Pi_{\theta x}^T & -\Pi_{xx} \\ -\Pi_{xx} & -\Pi_{\theta \theta} \end{bmatrix} ; \epsilon \right)
\]

is stable.

\(^{17}\)A solution \(X\) of the matrix Riccati equation \(\dot{X} + XA + XBB^TX - CC^T = 0\) is called stabilizing if \(-A + BB^TX\) is a stable matrix, see \(\text{Zhou et al.} 1996\) Chapter 13 for more details.
Proof. Fix $\epsilon > 0$ and an arbitrary matrix $\Pi_{\theta x} \in \mathbb{R}^{m \times d}$ (note that this is not necessarily a solution of the coupled Equation [B.5] but an arbitrary given matrix). When $\Pi_{\theta x}$ is fixed, a standard result on autonomous matrix Riccati equations (Zhou et al., 1996, Theorem 13.7), shows that there exist unique stabilizing solutions $\Pi_{\theta \theta}$ and $\Pi_{xx}$ of the decoupled equations (B.3) and (B.4), that are also automatically positive semidefinite.

Set
\[
y = \Lambda^{-1/2} \Pi_{\theta x} (\gamma \sigma_x \sigma_x^T)^{1/2}, \quad y_1 = (\gamma \sigma_x \sigma_x^T)^{1/2} \Pi_{xx} (\gamma \sigma_x \sigma_x^T)^{1/2}, \quad y_3 = \Lambda^{-1/2} \Pi_{\theta \theta} \Lambda^{-1/2}.
\]

Then by (Zhou et al., 1996, Theorem 13.7), the matrices $y_1 \in \mathbb{R}^{d \times d}$ and $y_3 \in \mathbb{R}^{m \times m}$ are the unique stabilizing (and in turn also positive semidefinite) solutions of the following equations:
\[
y_1^2 + \left( \frac{r}{2} I_d - (\sigma_x \sigma_x^T)^{-1/2} A_x (\sigma_x \sigma_x^T)^{-1/2} \right)^T y_1 + y_1 \left( \frac{r}{2} I_d - (\sigma_x \sigma_x^T)^{-1/2} A_x (\sigma_x \sigma_x^T)^{-1/2} \right) = \epsilon y^T y, \quad (B.25)
\]
\[
\epsilon y_3^2 + r y_3 = \gamma \left( y (\gamma \sigma_x \sigma_x^T)^{-1/2} \sigma_x - \Lambda^{-1/2} \sigma_p \right) \left( y (\gamma \sigma_x \sigma_x^T)^{-1/2} \sigma_x - \Lambda^{-1/2} \sigma_p \right)^T. \quad (B.26)
\]

Moreover, there exists a unique stabilizing solution $y_2 \in \mathbb{R}^{d \times d}$ of the coupled equation
\[
y_2^2 - \left( \frac{r}{2} I_d - (\sigma_x \sigma_x^T)^{-1/2} A_x (\sigma_x \sigma_x^T)^{-1/2} \right)^T y_2 - y_2 \left( \frac{r}{2} I_d - (\sigma_x \sigma_x^T)^{-1/2} A_x (\sigma_x \sigma_x^T)^{-1/2} \right) = 0. \quad (B.27)
\]

Since $A_x - \frac{r}{2} I_d$ is stable, $(\sigma_x \sigma_x^T)^{-1/2} A_x (\sigma_x \sigma_x^T)^{-1/2} - \frac{r}{2} I_d$ is stable as well. Thus, by verifying the conditions in (Zhou et al., 1996, Theorem 13.7), it follows that $y_2$ is positive definite. As a consequence, $y_1 + y_2$ and $\epsilon y_3 + \frac{r}{2} I_m$ are positive definite as well as sums of positive definite matrices and positive semidefinite matrices.

Since similar matrices have the same eigenvalues, the stability of $\mathcal{A} \left( \begin{bmatrix} \Pi_{xx} & -\Pi_{\theta x} \\ -\Pi_{\theta x} & \Pi_{\theta \theta} \end{bmatrix}, \epsilon \right)$ is equivalent to the stability of
\[
\bar{\mathcal{A}} = \begin{bmatrix} (\gamma \sigma_x \sigma_x^T)^{-1/2} & 0 \\ 0 & \Lambda^{1/2} / \sqrt{\epsilon} \end{bmatrix} \mathcal{A} \left( \begin{bmatrix} \Pi_{xx} & -\Pi_{\theta x} \\ -\Pi_{\theta x} & \Pi_{\theta \theta} \end{bmatrix}, \epsilon \right) \begin{bmatrix} (\gamma \sigma_x \sigma_x^T)^{-1/2} & 0 \\ 0 & \sqrt{\epsilon} \Lambda^{-1/2} \end{bmatrix}
\]
\[
= \begin{bmatrix} \sqrt{\epsilon} y_1 + \frac{r}{2} I_d - (\sigma_x \sigma_x^T)^{-1/2} A_x (\sigma_x \sigma_x^T)^{-1/2} & \sqrt{\epsilon} \left( \sqrt{\epsilon} \Lambda^{-1/2} \sigma_p \sigma_x^T (\sigma_x \sigma_x^T)^{-1/2} - y \right)^T \\ \sqrt{\epsilon} \left( \epsilon y_3 + \frac{r}{2} I_m \right) y & \epsilon y_3 + \frac{r}{2} I_m \end{bmatrix}.
\]

In view of the Lyapunov Lemma as in (Zhou et al., 1996, Lemma 3.18), it therefore suffices to show the positive definiteness of
\[
\begin{bmatrix} y_1 + y_2 & 0 \\ 0 & \epsilon y_3 + \frac{r}{2} I_m \end{bmatrix} \bar{\mathcal{A}} + \bar{\mathcal{A}}^T \begin{bmatrix} y_1 + y_2 & 0 \\ 0 & \epsilon y_3 + \frac{r}{2} I_m \end{bmatrix},
\]

where
\[
\begin{bmatrix} y_1 + y_2 & 0 \\ 0 & \epsilon y_3 + \frac{r}{2} I_m \end{bmatrix} \bar{\mathcal{A}}
\]
\[
= \left( y_1 + y_2 \right) \left( \sqrt{\epsilon} y_1 + y_2 \right) \left( \sqrt{\epsilon} \Lambda^{-1/2} \sigma_p \sigma_x^T (\sigma_x \sigma_x^T)^{-1/2} - y \right)^T.
\]
To this end, first notice that, by definition of $y_3$ and $y$,

\[
\left( ey_3 + \frac{r}{2} I_m \right)^2 = \frac{r^2}{4} I_m + e \gamma \left( y (\gamma \sigma_x \sigma_x^T)^{-1/2} \sigma_x - \Lambda^{-1/2} \sigma_p \right) \left( y (\gamma \sigma_x \sigma_x^T)^{-1/2} \sigma_x - \Lambda^{-1/2} \sigma_p \right)^T
\]

\[
= \frac{r^2}{4} I_m + e \left( (yy^T - \sqrt{\gamma} \Lambda^{-1/2} \sigma_p \sigma_x^T (\sigma_x \sigma_x^T)^{-1/2} y^T - \sqrt{\gamma} y (\sigma_x \sigma_x^T)^{-1/2} \sigma_p \Lambda^{-1/2} \sigma_x \sigma_x^T)^T \right)
\]

\[
+ e \gamma \Lambda^{-1/2} \sigma_p \sigma_x^T (\sigma_x \sigma_x^T)^{-1} \sigma_x \sigma_x^T \Lambda^{-1/2} + e \gamma \Lambda^{-1/2} \sigma_p \left( I_k - \sigma_x^T (\sigma_x \sigma_x^T)^{-1} \sigma_x \right) \sigma_p \Lambda^{-1/2}
\]

\[
= \frac{r^2}{4} I_m + e \gamma \Lambda^{-1/2} \sigma_p \left( I_k - \sigma_x^T (\sigma_x \sigma_x^T)^{-1} \sigma_x \right) \sigma_p \Lambda^{-1/2}
\]

\[
> e \gamma \Lambda^{-1/2} (\sigma_p \sigma_x^T (\sigma_x \sigma_x^T)^{-1/2} y) \left( y - \sqrt{\gamma} \Lambda^{-1/2} \sigma_p \sigma_x^T (\sigma_x \sigma_x^T)^{-1/2} \right)^T.
\]

(B.28)

Next, the matrix equations (B.25) and (B.27) satisfied by $y_1$ and $y_2$ imply that

\[
(y_1 + y_2) \left( y_1 + \frac{r}{2} I_d - (\sigma_x \sigma_x^T)^{-1/2} A_x (\sigma_x \sigma_x^T)^{1/2} \right) + \left( y_1 + \frac{r}{2} I_d - (\sigma_x \sigma_x^T)^{-1/2} A_x (\sigma_x \sigma_x^T)^{1/2} \right)^T (y_1 + y_2)
\]

\[
= y_1^2 + y_2 y_1 + y_1 y_2 + y_2 \left( \frac{r}{2} I_d - (\sigma_x \sigma_x^T)^{-1/2} A_x (\sigma_x \sigma_x^T)^{1/2} \right) + \left( \frac{r}{2} I_d - (\sigma_x \sigma_x^T)^{-1/2} A_x (\sigma_x \sigma_x^T)^{1/2} \right)^T y_2
\]

\[
+ y_1^2 + y_1 \left( \frac{r}{2} I_d - (\sigma_x \sigma_x^T)^{-1/2} A_x (\sigma_x \sigma_x^T)^{1/2} \right) + \left( \frac{r}{2} I_d - (\sigma_x \sigma_x^T)^{-1/2} A_x (\sigma_x \sigma_x^T)^{1/2} \right)^T y_1
\]

\[
= y_1^2 + y_2 y_1 + y_1 y_2 + y_2^2 + ey^T y
\]

\[
= (y_1 + y_2)^2 + ey^T y > 0.
\]

(B.29)

This allows us to finally show the positive definiteness of

\[
\begin{bmatrix} y_1 + y_2 & 0 \\ 0 & ey_3 + \frac{r}{2} I_m \end{bmatrix} \Lambda + \Lambda^T \begin{bmatrix} y_1 + y_2 & 0 \\ 0 & ey_3 + \frac{r}{2} I_m \end{bmatrix}.
\]

To this end, for every $x \in \mathbb{R}^d$ and $\theta \in \mathbb{R}^m$, first notice that

\[
\begin{bmatrix} x^T \\ \theta^T \end{bmatrix} \left( \begin{bmatrix} y_1 + y_2 & 0 \\ 0 & ey_3 + \frac{r}{2} I_m \end{bmatrix} \Lambda + \Lambda^T \begin{bmatrix} y_1 + y_2 & 0 \\ 0 & ey_3 + \frac{r}{2} I_m \end{bmatrix} \right) \begin{bmatrix} x \\ \theta \end{bmatrix}
\]

\[
= 2\sqrt{\epsilon} \theta^T \left( \left( ey_3 + \frac{r}{2} I_m \right) y + \left( \sqrt{\gamma} \Lambda^{-1/2} \sigma_p \sigma_x^T (\sigma_x \sigma_x^T)^{-1/2} - y \right) (y_1 + y_2) \right) x
\]

\[
+ x^T \left( (y_1 + y_2)^2 + ey^T y \right) x + 2\theta^T \left( ey_3 + \frac{r}{2} I_m \right)^2 \theta
\]

\[
\geq x^T \left( (y_1 + y_2)^2 + ey^T y \right) x + 2\theta^T \left( ey_3 + \frac{r}{2} I_m \right)^2 \theta - 2 \left\| (ey_3 + \frac{r}{2} I_m) \theta \right\| \left\| (ey_3 + \frac{r}{2} I_m) \right\|
\]

\[
- 2 \left\| \sqrt{\epsilon} \left( \sqrt{\gamma} \Lambda^{-1/2} \sigma_p \sigma_x^T (\sigma_x \sigma_x^T)^{-1/2} - y \right)^T \theta \right\| \left\| (y_1 + y_2) x \right\|.
\]

Then by (B.28), for every $\theta \in \mathbb{R}^m$,

\[
\left\| \sqrt{\epsilon} \left( \sqrt{\gamma} \Lambda^{-1/2} \sigma_p \sigma_x^T (\sigma_x \sigma_x^T)^{-1/2} - y \right)^T \theta \right\| \leq \left\| (ey_3 + \frac{r}{2} I_m) \theta \right\|.
\]
with equality if and only if \( \theta = 0 \). Hence, for \( \theta \neq 0 \),

\[
\begin{bmatrix} x \\ \theta \end{bmatrix}^\top \left( \begin{bmatrix} y_1 + y_2 & 0 \\ 0 & \epsilon y_3 + \frac{r}{2} I_m \end{bmatrix} \bar{A} + \bar{A}^\top \begin{bmatrix} y_1 + y_2 & 0 \\ 0 & \epsilon y_3 + \frac{r}{2} I_m \end{bmatrix} \right) \begin{bmatrix} x \\ \theta \end{bmatrix}
\]

\[
> x^\top \left( (y_1 + y_2)^2 + \epsilon y^\top y \right) x + 20^\top \left( \epsilon y_3 + \frac{r}{2} I_m \right)^2 \theta - 2 \left\| \left( \epsilon y_3 + \frac{r}{2} I_m \right) \theta \right\| \left( \|\sqrt{\epsilon} x\| + \|(y_1 + y_2)x\| \right)
\]

\[
= \left( \left\| \left( \epsilon y_3 + \frac{r}{2} I_m \right) \theta \right\| - \|\sqrt{\epsilon} x\| \right)^2 + \left( \left\| \left( \epsilon y_3 + \frac{r}{2} I_m \right) \theta \right\| - \|(y_1 + y_2)x\| \right)^2 \geq 0.
\]

For \( \theta = 0 \), by (3.29),

\[
\begin{bmatrix} x \\ 0 \end{bmatrix}^\top \left( \begin{bmatrix} y_1 + y_2 & 0 \\ 0 & \epsilon y_3 + \frac{r}{2} I_m \end{bmatrix} \bar{A} + \bar{A}^\top \begin{bmatrix} y_1 + y_2 & 0 \\ 0 & \epsilon y_3 + \frac{r}{2} I_m \end{bmatrix} \right) \begin{bmatrix} x \\ 0 \end{bmatrix}
\]

\[
= x^\top \left( (y_1 + y_2)^2 + \epsilon y^\top y \right) x \geq 0.
\]

In summary, together with the positive semidefiniteness of \((y_1 + y_2)^2 + \epsilon y^\top y\), it follows that

\[
\begin{bmatrix} x \\ \theta \end{bmatrix}^\top \left( \begin{bmatrix} y_1 + y_2 & 0 \\ 0 & \epsilon y_3 + \frac{r}{2} I_m \end{bmatrix} \bar{A} + \bar{A}^\top \begin{bmatrix} y_1 + y_2 & 0 \\ 0 & \epsilon y_3 + \frac{r}{2} I_m \end{bmatrix} \right) \begin{bmatrix} x \\ \theta \end{bmatrix} = 0
\]

if and only if \( \theta = 0 \) and \( x = 0 \). Therefore, Lyapunov’s Lemma as in (Zhou et al., 1996, Lemma 3.19) yields the asserted stability of \(-A \left( \begin{bmatrix} \Pi_{xx} & -\Pi_{\theta x} \\ -\Pi_{\theta x} & -\Pi_{\theta \theta} \end{bmatrix} ; \sigma \right)\). \(\square\)

### C Proof of Theorem 3.1(ii)

We now show that the linear equations in Theorem 3.1(ii) also have a unique solution. To this end, first recall that the solution \(\Pi_{xx}\) of (3.2) is positive semidefinite, so that \(-\left( \frac{r}{2} I_d - A_x + \gamma \sigma_x \sigma_x^\top \Pi_{xx} \right)\) is stable and \(-\left( r I_d - A_x + \gamma \sigma_x \sigma_x^\top \Pi_{xx} \right)\) is in turn stable as well and thus invertible. As a consequence, (3.5) can be solved for \(\Pi_x\) in terms of \(\Pi_{\theta}\):

\[
\Pi_x = -\left( r I_d - A_x^\top + \gamma \Pi_{xx} \left( \sigma_x \sigma_x^\top \right) \right)^{-1} \Pi_{\theta x} A^{-1} \Pi_{\theta},
\]

After plugging this result into (3.4) and multiplying by \(\Lambda^{-1/2}\), it therefore remains to show that \(\Pi_{\theta}\) is well defined as the solution of the following autonomous linear system:

\[
-\Lambda^{-1/2} \dot{\Pi}_{\theta}
\]

\[
= \left( \Lambda^{-1/2} \left( (\Pi_{\theta x} \sigma_x - \sigma_x) \sigma_x^\top \left( r I_d - A_x^\top + \gamma \Pi_{xx} \sigma_x \sigma_x^\top \right)^{-1} \Pi_{\theta x} + \Pi_{\theta \theta} \right) \Lambda^{-1/2} + r I_m \right) \Lambda^{-1/2} \Pi_{\theta}.
\]

To this end, we need to show that the matrix multiplying \(\Lambda^{-1/2} \Pi_{\theta}\) is invertible.

We first focus on the relationship between \(\frac{r}{2} I_d - A_x + \gamma \sigma_x \sigma_x^\top \Pi_{xx}\) and \(\Pi_{\theta x}\) and rewrite (3.2) as

\[
\left( \frac{r}{2} I_d - A_x^\top + \gamma \Pi_{xx} \sigma_x \sigma_x^\top \right) \left( \gamma \sigma_x \sigma_x^\top \right)^{-1} \left( \frac{r}{2} I_d - A_x + \gamma \sigma_x \sigma_x^\top \Pi_{xx} \right)
\]

\[
= \Pi_{\theta x} \Lambda^{-1} \Pi_{\theta x} + \left( \frac{r}{2} I_d - A_x \right)^\top \left( \gamma \sigma_x \sigma_x^\top \right)^{-1} \left( \frac{r}{2} I_d - A_x \right).
\]
As a consequence,
\[
\left\| \left( \gamma \sigma_x \sigma_x^\top \right)^{1/2} \left( r I_d - A_x + \gamma \Pi_{xx} \sigma_x \sigma_x^\top \right)^{-1} \Pi_{\theta x} \Lambda^{-1/2} \right\|^2 \\
= \left\| \left( \gamma \sigma_x \sigma_x^\top \right)^{1/2} \left( r I_d - A_x + \gamma \Pi_{xx} \sigma_x \sigma_x^\top \right)^{-1} \Pi_{\theta x} \Lambda^{-1/2} \Pi_{\theta x} \left( r I_d - A_x + \gamma \sigma_x \sigma_x^\top \Pi_{xx} \right)^{-1} \left( \gamma \sigma_x \sigma_x^\top \right)^{1/2} \right\|^2 \\
\leq \left\| \left( \gamma \sigma_x \sigma_x^\top \right)^{1/2} \left( r I_d - A_x + \gamma \Pi_{xx} \sigma_x \sigma_x^\top \right)^{-1} \left( \frac{r}{2} I_d - A_x + \gamma \Pi_{xx} \sigma_x \sigma_x^\top \right) \left( \gamma \sigma_x \sigma_x^\top \right)^{-1/2} \right\|^2 \leq 1
\]
and in turn
\[
\left\| \sigma_x^\top \left( r I_d - A_x + \gamma \Pi_{xx} \sigma_x \sigma_x^\top \right)^{-1} \Pi_{\theta x} \Lambda^{-1/2} \right\| \\
= \left\| \sigma_x^\top \left( \gamma \sigma_x \sigma_x^\top \right)^{-1/2} \left( \gamma \sigma_x \sigma_x^\top \right)^{1/2} \left( r I_d - A_x + \gamma \Pi_{xx} \sigma_x \sigma_x^\top \right)^{-1} \Pi_{\theta x} \Lambda^{-1/2} \right\| \\
\leq \frac{1}{\sqrt{\gamma}} \left\| \left( \gamma \sigma_x \sigma_x^\top \right)^{1/2} \left( r I_d - A_x + \gamma \Pi_{xx} \sigma_x \sigma_x^\top \right)^{-1} \Pi_{\theta x} \Lambda^{-1/2} \right\| \leq \frac{1}{\sqrt{\gamma}}. \tag{C.1}
\]

Similarly, we can rewrite (3.1) as
\[
\gamma \Lambda^{-1/2} (\Pi_{\theta x} \sigma_x - \sigma_p) (\Pi_{\theta x} \sigma_x - \sigma_p)^\top \Lambda^{-1/2} + \frac{r^2}{4} I_m = \left( \Lambda^{-1/2} \Pi_{\theta \theta} \Lambda^{-1/2} + \frac{r^2}{2} I_m \right)^2.
\]

By singular value decomposition, there exists a \( k \times k \) orthonormal matrix \( U \) such that
\[
\sqrt{\gamma} \Lambda^{-1/2} (\Pi_{\theta x} \sigma_x - \sigma_p) = \left[ \left( \left( \Lambda^{-1/2} \Pi_{\theta \theta} \Lambda^{-1/2} + \frac{r^2}{2} I_m \right)^2 - \frac{r^2}{4} I_m \right)^{1/2} \right] U. \tag{C.2}
\]

Hence, for arbitrary \( x \in \mathbb{R}^d \), using first the triangle inequality, then \( (C.1) \) and finally \( (C.2) \):
\[
\left\| x^\top \left( \Lambda^{-1/2} \left( \Pi_{\theta x} \sigma_x - \sigma_p \right) \sigma_x^\top \left( r I_d - A_x + \gamma \Pi_{xx} \left( \sigma_x \sigma_x^\top \right)^{-1} \Pi_{\theta x} + \Pi_{\theta \theta} \right) \Lambda^{-1/2} + r I_m \right) \right\| \\
\geq \left\| x^\top \left( \Lambda^{-1/2} \Pi_{\theta \theta} \Lambda^{-1/2} + r I_m \right) \right\| - \left\| x^\top \Lambda^{-1/2} (\Pi_{\theta x} \sigma_x - \sigma_p) \right\| \\
= \left\| x^\top \left( \Lambda^{-1/2} \Pi_{\theta \theta} \Lambda^{-1/2} + r I_m \right) \right\| - \left\| x^\top \left( \left( \Lambda^{-1/2} \Pi_{\theta \theta} \Lambda^{-1/2} + \frac{r^2}{2} I_m \right)^2 - \frac{r^2}{4} I_m \right)^{1/2} \right\| \\
= \frac{x^\top \left( \Lambda^{-1/2} \Pi_{\theta \theta} \Lambda^{-1/2} + \frac{r^2}{2} I_m \right)^2 x + \frac{r^2}{2} \| x \|^2}{\left\| x^\top \left( \Lambda^{-1/2} \Pi_{\theta \theta} \Lambda^{-1/2} + r I_m \right) \right\| + \left\| x^\top \left( \left( \Lambda^{-1/2} \Pi_{\theta \theta} \Lambda^{-1/2} + \frac{r^2}{2} I_m \right)^2 - \frac{r^2}{4} I_m \right)^{1/2} \right\|} \\
\geq \frac{r^2 \| x \|^2}{2 \left\| \Lambda^{-1/2} \Pi_{\theta \theta} \Lambda^{-1/2} + r I_m \right\| \| x \|} \\
= \frac{r^2 \| x \|^2}{2r + 2 \left\| \Lambda^{-1/2} \Pi_{\theta \theta} \Lambda^{-1/2} \right\|}. 
\]
Therefore, the smallest singular value $\sigma_{\min}$ of the matrix is strictly positive as
\[
\sigma_{\min} \left( \Lambda^{-1/2} \left( (\Pi_{\theta x} \sigma_x - \sigma_p) \sigma_x^\top (rI_d - A_x + \gamma \Pi_{xx} (\sigma_x \sigma_x^\top))^{-1} \Pi_{\theta x}^\top + \Pi_{\theta \theta} \right) \Lambda^{-1/2} + rI_m \right)
\]
\[
= \min_{x \neq 0} \left\{ \left\| \frac{x}{x^\top} \left( \Lambda^{-1/2} \left( (\Pi_{\theta x} \sigma_x - \sigma_p) \sigma_x^\top (rI_d - A_x + \gamma \Pi_{xx} (\sigma_x \sigma_x^\top))^{-1} \Pi_{\theta x}^\top + \Pi_{\theta \theta} \right) \Lambda^{-1/2} + rI_m \right) \right\| \right\}
\]
\[
\geq \frac{r^2}{2r + 2 \left\| \Lambda^{-1/2} \Pi_{\theta \theta} \Lambda^{-1/2} \right\|} > 0.
\]
This matrix therefore is invertible, and a unique solution $\Pi_{\theta}$, $\Pi_x$ of (3.4)–(3.5) thus indeed exists.

**D  Proof of Theorem 3.1(iii)**

With the candidate [3.7] at hand, we now present a verification theorem, which shows that it indeed identifies the value function for the lifetime consumption problem (2.4) and that the corresponding feedback controls (3.8), (3.9) are optimal. To ease notation, we use the shorthand notation
\[
J(t, x, w, \theta) = e^{-\delta t} J(x, w, \theta).
\]
In order to rule out doubling strategies and excessive borrowing ("Ponzi schemes"), we focus on admissible policies $c_t$, $\theta_t$ for which (i) the local martingale
\[
\gamma \int_0^\tau J(t, X(t), W^{c, \theta}(t), \theta^{c, \theta}(t)) \left( (\Pi_{\theta x} \sigma_x - \sigma_p)^\top \theta(t) - \sigma_x^\top (\Pi_{xx} X(t) - \Pi_x) \right) dt
\]
\[
is a true martingale and (ii) the following transversality condition holds:
\[
\lim_{\tau \to \infty} \mathbb{E} \left[ J(\tau, X(\tau), W^{c, \theta}(\tau), \theta^{c, \theta}(\tau)) \right] \to 0. \tag{D.2}
\]
Both of these conditions are satisfied if the risky positions and the negative part of the agent’s wealth are integrable enough (which rules out doubling strategies and Ponzi schemes). In particular, both (D.1) and (D.2) hold for controls for which the corresponding risky positions are uniformly bounded and wealth is bounded from below. With substantially more sophisticated estimates, we can also show that our candidate policy (3.8), (3.9) also satisfies these requirements, see Appendix D.1.

Let $(c, \tilde{\theta})$ be any admissible policy. For any $\tau > 0$, Itô’s formula yields
\[
J(\tau, X(\tau), W^{c, \theta}(\tau), \theta^{c, \theta}(\tau))
\]
\[
= J(0, x_0, w_0, \theta_0) + \int_0^\tau \left( \partial_t J + \mathcal{L}^{c, \theta} J \right) dt
\]
\[
+ \gamma \int_0^\tau J(t, X(t), W^{c, \theta}(t), \theta^{c, \theta}(t)) \left( (\Pi_{\theta x} \sigma_x - \sigma_p)^\top \theta(t) - \sigma_x^\top (\Pi_{xx} X(t) - \Pi_x) \right) dB(t),
\]
where
\[
\mathcal{L}^{c, \theta} J = \theta^\top \sigma_p \sigma_p^\top \frac{\partial^2 J}{\partial w \partial \theta} + \frac{1}{2} \text{tr} \left( \sigma_x \sigma_x^\top \frac{\partial^2 J}{\partial x^2} \right) + x^\top A_x^\top \frac{\partial J}{\partial x} + \frac{1}{2} \sigma_p \sigma_p^\top \theta \frac{\partial^2 J}{\partial w^2} + \theta^\top \frac{\partial J}{\partial \theta}
\]
\[
+ \left( rw + y - c + \theta^\top (\bar{\mu} + C_{xx}) - \frac{1}{2} \theta^\top \Lambda \theta \right) \frac{\partial J}{\partial w}.
\]
For admissible policies, the stochastic integral is a true martingale and thus has expectation zero. As a consequence,  
\[ E \left[ J(\tau, X(\tau), W^{c,\hat{\theta}}(\tau), \theta^{c,\hat{\theta}}(\tau)) \right] \]

\[ = E \left[ \int_0^\tau \left( \partial_t J + L^{c,\hat{\theta}} J \right) \left( t, X(t), W^{c,\hat{\theta}}(t), \theta^{c,\hat{\theta}}(t) \right) dt \right] + J(t, x_0, w_0, \theta_0) \]

\[ \geq E \left[ \int_0^\tau e^{-\delta t - \beta c(t)} dt \right] + J(0, x_0, w_0, \theta_0), \]

where the inequality holds because \( J \) solves the HJB equation (A.1). For \( \tau \to \infty \), the transversality condition (D.2) and monotone convergence in turn yield  
\[ J(0, x_0, w_0, \theta_0) \geq E \left[ \int_0^\infty -e^{-\delta t - \beta c(t)} dt \right]. \]

This inequality becomes an equality for the pointwise maximizers (3.8),(3.9) of the HJB equation, if we can show that the corresponding feedback control is admissible. This is established in Appendix D.1.

**D.1 Admissibility of the Optimal Control in Theorem 3.1(iii)**

In this part, we establish the most difficult element for the proof of Theorem 3.1(iii): the admissibility of the feedback controls (3.8), (3.9). The first step is to use the Riccati equations for the coefficients of the candidate value function (3.7), the definitions of the controls (3.8), (3.9), and tedious but elementary algebraic manipulations to obtain the following compact representation for the value function evaluated along the candidate controls. (Since only these controls appear in the present section, we drop the corresponding indices to ease notation.)

**Proposition D.1.** For any \( t > 0 \), we have

\[ J(t, X(t), W(t), \theta(t)) = J(0, x_0, w_0, \theta_0) \exp \left( -rt + \frac{\gamma^2}{2} \langle M \rangle_t \right), \tag{D.3} \]

for the local martingale

\[ dM_t = \left( \Pi_{\theta x} \sigma_x - \sigma_p \right)^\top \theta(t) - \sigma_x^\top (\Pi_{xx} X(t) - \Pi_x) dB(t). \tag{D.4} \]

**Proof.** After inserting the controls (3.8), (3.9), the corresponding wealth dynamics can be rewritten
as

\[ dW(t) = \left( \frac{r}{\gamma} \ln r J_0 + r \left( -\frac{1}{2} \ln r J_0 \right)^T \Pi_x X(t) + \Pi_x^T X(t) + \frac{1}{2} \theta(t)^T \Pi_{\theta\theta} \theta(t) + \Pi_{\theta}^T \theta(t) + \theta(t)^T \Pi_{\theta x} X(t) \right) dt \]

\[ + \left( \theta(t)^T (\bar{\mu} + C_x X(t)) - \frac{1}{2} (\Pi_{\theta\theta} \theta(t) + \Pi_{\theta x} X(t) + \Pi_{\theta})^T \Lambda^{-1} (\Pi_{\theta\theta} \theta(t) + \Pi_{\theta x} X(t) + \Pi_{\theta}) \right) dt \]

\[ + d \int_0^t e^{r(t-u)} y(u) du + \theta(t)^T \sigma_p dB(t) \]

\[ = \left( \theta(t)^T (\bar{\mu} + C_x X(t)) - \frac{1}{2} (\Pi_{\theta\theta} \theta(t) + \Pi_{\theta x} X(t) + \Pi_{\theta})^T \Lambda^{-1} (\Pi_{\theta\theta} \theta(t) + \Pi_{\theta x} X(t) + \Pi_{\theta}) \right) dt \]

\[ + r \left( -\frac{1}{2} \ln r J_0 \right)^T \Pi_x X(t) + \Pi_x^T X(t) + \frac{1}{2} \theta(t)^T \Pi_{\theta\theta} \theta(t) + \Pi_{\theta}^T \theta(t) + \theta(t)^T \Pi_{\theta x} X(t) \right) dt \]

\[ + \frac{1}{\gamma} \left( r - \delta - \frac{\gamma}{2} \Pi_{\theta x} \Lambda^{-1} \Pi_{\theta} + \frac{\gamma^2}{2} \Pi_x^T \sigma_x \sigma_x^T \Pi_x - \gamma \right) \left( \sigma_x^T \Pi_{\theta x} \sigma_x \right) dt \]

\[ + d \int_0^t e^{r(t-u)} y(u) du + \theta(t)^T \sigma_p dB(t). \] (D.5)

Next, we calculate the dynamics for the other terms appearing in the exponential of \( J \), cf. (3.7). By definition of the trading rate (3.9), we have

\[ d\theta(t)^T \Pi_{\theta\theta} \theta(t) = -2 \theta(t)^T \Pi_{\theta\theta} \Lambda^{-1} (\Pi_{\theta\theta} \theta(t) + \Pi_{\theta x} X(t) + \Pi_{\theta}) dt. \] (D.6)

Likewise, the dynamics of the state process yield

\[ dX(t)^T \Pi_{xx} X(t) = 2X(t)^T \Pi_{xx} A_x X(t) dt + 2X(t)^T \Pi_{xx} \sigma_x dB(t) + \text{tr} \left( \sigma_x^T \Pi_{xx} \sigma_x \right) dt, \] (D.7)

as well as

\[ d\theta(t)^T \Pi_{\theta x} X(t) = \left( \theta(t)^T \Pi_{\theta x} A_x X(t) - (\Pi_{\theta\theta} \theta(t) + \Pi_{\theta x} X(t) + \Pi_{\theta})^T \Lambda^{-1} \Pi_{\theta x} X(t) \right) dt \]

\[ + \theta(t)^T \Pi_{\theta x} \sigma_x dB(t). \] (D.8)

Finally,

\[ d\Pi_{\theta} \theta(t) = -\Pi_{\theta} \Lambda^{-1} (\Pi_{\theta\theta} \theta(t) + \Pi_{\theta x} X(t) + \Pi_{\theta}) dt, \] (D.9)

as well as

\[ d\Pi_{\theta x} X(t) = \Pi_{\theta x} A_x X(t) dt + \Pi_x \sigma_x dB(t). \] (D.10)
Putting together (D.5)-(D.10), we obtain

\[
\begin{align*}
& d \left( \int_0^t e^{r(t-u)} y(u) du - W(t) - \frac{1}{2} X(t)^\top \Pi_{xx} X(t) + \Pi_x^\top X(t) + \frac{1}{2} \theta(t)^\top \Pi_{\theta\theta} \theta(t) + \Pi_\theta^\top \theta(t) + \theta(t)^\top \Pi_{\theta x} X(t) \right) \\
& = \left( \Pi_{\theta x}^\top \theta(t) + \Pi_x - \Pi_{xx} X(t) \right)^\top A_x X(t) - \frac{1}{2} \left( \Pi_{\theta\theta} \theta(t) + \Pi_{\theta x} X(t) + \Pi_\theta \theta(t) + \theta(t)^\top \Pi_{\theta x} X(t) \right) \right) dt \\
& - r \left( \theta(t)^\top (\bar{\mu} + C_x X(t)) - \frac{1}{2} X(t)^\top \Pi_{xx} X(t) + \Pi_x^\top X(t) + \frac{1}{2} \theta(t)^\top \Pi_{\theta\theta} \theta(t) + \Pi_\theta^\top \theta(t) + \theta(t)^\top \Pi_{\theta x} X(t) \right) \right) dt \\
& + \left( \left( \bar{\rho} - \frac{\gamma}{2} - \Pi_{\theta x}^\top \Lambda^{-1} \Pi_{\theta x} - \frac{1}{2} \Pi_x^\top \sigma_x \sigma_x^\top \Pi_x \right) dt + \left( \sigma_x^\top (\Pi_x + \Pi_{\theta x} \theta(t) - \Pi_{xx} X(t)) - \sigma_p^\top \theta(t) \right)^\top dB(t) \right) \\
& = - \left( \frac{1}{2} \theta(t)^\top (\Pi_{\theta\theta} \Lambda^{-1} \Pi_{\theta x} + r\Pi_{\theta x}) \theta(t) + \frac{1}{2} X(t)^\top \left( \Pi_{\theta x}^\top \Lambda^{-1} \Pi_{\theta x} + \Pi_{xx} A_x + A_x^\top \Pi_{xx} - r\Pi_{xx} \right) X(t) \right) \right) dt \\
& - \left( \theta(t)^\top (\Pi_{\theta\theta} \Lambda^{-1} \Pi_{\theta x} + C_x - \Pi_{\theta x} A_x) X(t) + \frac{\bar{\rho} - \frac{\gamma}{2} - \Pi_x^\top \sigma_x \sigma_x^\top \Pi_x \right) dt \\
& - \left( \left( rI_d - A_x \right)^\top \Pi_x + \Pi_{\theta x}^\top \Lambda^{-1} \Pi_\theta \right)^\top X(t) + \left( \left( \Lambda^{-1} \Pi_{\theta\theta} + rI_m \right)^\top \Pi_\theta + \bar{\mu} \right)^\top \theta(t) \right) dt \\
& + \left( \left( \Pi_{\theta x} \sigma_x - \sigma_p \right)^\top \theta(t) - \sigma_x^\top (\Pi_{xx} X(t) - \Pi_x) \right) \right) dB(t). \quad \text{(D.11)}
\end{align*}
\]

Using the Riccati equation (3.1) for \( \Pi_{\theta\theta} \), the term quadratic term in \( \theta \) can be rewritten as

\[
\frac{1}{2} \theta(t)^\top \left( \Pi_{\theta\theta} \Lambda^{-1} \Pi_{\theta x} + r\Pi_{\theta x} \right) \theta(t) = \frac{\gamma}{2} \theta(t)^\top (\Pi_{\theta x} \sigma_x - \sigma_p) (\Pi_{\theta x} \sigma_x - \sigma_p)^\top \theta(t).
\]

Likewise, in view of the Riccati equation (3.2) for \( \Pi_{xx} \), the term quadratic in \( X \) becomes:

\[
\frac{1}{2} X(t)^\top \left( \Pi_{\theta x}^\top \Lambda^{-1} \Pi_{\theta x} + \Pi_{xx} A_x + A_x^\top \Pi_{xx} - r\Pi_{xx} \right) X(t) = \frac{\gamma}{2} X(t)^\top \Pi_{xx} \sigma_x \sigma_x^\top \Pi_{xx} X(t),
\]

With the equation (3.3) for \( \Pi_{\theta x} \), the cross term for \( \theta \) and \( X \) can be rewritten as

\[
\theta(t)^\top \left( \Pi_{\theta\theta} \Lambda^{-1} \Pi_{\theta x} + C_x + r\Pi_{\theta x} - \Pi_{\theta x} A_x \right) X(t) = -\gamma \theta(t)^\top (\Pi_{\theta x} \sigma_x - \sigma_p) \sigma_x^\top \Pi_{xx} X(t).
\]

Finally, in view of the equations (3.4) and (3.5) for \( \Pi_\theta \) and \( \Pi_x \), the terms linear in \( \theta \) and \( X \) become

\[
\left( \left( \Lambda^{-1} \Pi_{\theta\theta} + rI_m \right)^\top \Pi_\theta + \bar{\mu} \right)^\top \theta(t) = \gamma \Pi_x^\top \sigma_x \left( \Pi_{\theta x} \sigma_x - \sigma_p \right)^\top \theta(t).
\]

Altogether, we obtain

\[
\begin{align*}
& d\gamma(t) - W(t) + \int_0^t e^{r(t-u)} y(u) du - \frac{1}{2} X(t)^\top \Pi_{xx} X(t) + \Pi_x^\top X(t) \\
& + \frac{1}{2} \theta(t)^\top \Pi_{\theta\theta} \theta(t) + \Pi_\theta^\top \theta(t) + \theta(t)^\top \Pi_{\theta x} X(t) \right) \right) + \gamma dM_t - \frac{\gamma^2}{2} d\langle M \rangle_t + (\delta - r) dt.
\end{align*}
\]

For any \( \tau > 0 \), we therefore indeed have (D.3) as asserted.

The crucial step left to establish the admissibility of (3.8), (3.9) now is to show that the exponential local martingale \( \exp \left( M_t - \frac{1}{2} \langle M \rangle_t \right) \) is a true martingale. To wit, suppose this holds,
so that this process has expectation 1. The representation \([D.3]\) then immediately yields the
transversality condition \([D.2]\):

\[
\mathbb{E} \left[ J(\tau, X(\tau), W(\tau), \theta(\tau)) \right] = J(0, x_0, w_0, \theta_0) e^{-r\tau} \rightarrow 0 \quad \text{as} \quad \tau \to \infty.
\]

To establish admissibility, it then remains to verify the martingale property of \([D.1]\). In view
of \([D.3]\) and Itô’s formula,

\[
\int_0^\tau J(t, X(t), W(t), \theta(t)) dM_t
= J(\tau, X(\tau), W(\tau), \theta(\tau)) - J(0, x_0, w_0, \theta_0) + r \int_0^\tau J(t, X(t), W(t), \theta(t)) dt.
\]

If the process \(\exp(M_t - \frac{1}{2} \langle M \rangle_t)\) is a martingale, it follows from \([D.3]\) that this process is integrable
and, moreover, satisfies

\[
\mathbb{E}_t \left[ \int_t^\tau J(s, X(s), W(s), \theta(s)) dM_s \right] = \mathbb{E}_t \left[ J(\tau, X(\tau), W(\tau), \theta(\tau)) - J(t, X(t), W(t), \theta(t)) \right]
+ r \mathbb{E}_t \left[ \int_t^\tau J(s, X(s), W(s), \theta(s)) ds \right]
= J(t, X(t), W(t), \theta(t)) \left( e^{-r(\tau-t)} - 1 + r \int_t^\tau e^{-rs} ds \right)
= 0.
\]

(Here, we have again used \([D.3]\) and the martingale property of \(\exp(M_t - \frac{1}{2} \langle M \rangle_t)\) to compute the
conditional expectations in the next-to-last step.) Whence, the local martingale \([D.1]\) is indeed a
true martingale as required for the admissibility of the policy \((3.8), (3.9)\).

In summary, it therefore remains to show that the exponential local martingale \(\exp(M_t - \frac{1}{2} \langle M \rangle_t)\)
is indeed a true martingale. The first ingredient for establishing this is the following elementary
estimate for Gaussian processes:

**Lemma D.2.** Let \(\{X(t)\}_{t \geq 0}\) be an \(\mathbb{R}^d\)-valued Gaussian process with \(X(t) \sim N(0, \Sigma_t)\) and suppose
the covariance matrices \((\Sigma_t)_{t \geq 0}\) are uniformly bounded (\(\sup_{t \geq 0} \Sigma_t \leq \Sigma\) for a positive definite matrix
\(\Sigma)\). Then:

1. for every \(\alpha \in \mathbb{R}^{k \times d}\) and every \(t \geq 0\), there exists \(h_0 > 0\) such that, for every \(0 < h < h_0\):

\[
\mathbb{E} \left[ \exp \left( \frac{1}{2} \int_t^{t+h} \| \alpha^\top X(u) \|^2 du \right) \right] < \infty; \tag{D.12}
\]

2. for every \(\alpha \in \mathbb{R}^{k \times d}\), there exists \(h_1 > 0\) such that, for every \(0 < h < h_1\):

\[
\mathbb{E} \left[ \exp \left( \frac{1}{2} \int_{t_0}^{t_n} \int_0^v \| \alpha^\top X(u) \|^2 dudv \right) \right] < \infty, \tag{D.13}
\]

where \(t_0 = 0\) and \(t_n := \sum_{m=1}^n \frac{h}{m}\).
Proof. First, we show (D.12) holds. By the submultiplicativity of the matrix operator norm and the convexity of the exponential function, we have

\[
\mathbb{E}\left[ \exp\left( \frac{1}{2} \int_t^{t+h} \| \alpha^\top X(u) \|^2 \, du \right) \right] \leq \mathbb{E}\left[ \exp\left( \frac{\| \alpha \|^2}{2} \int_t^{t+h} \| X(u) \|^2 \, du \right) \right] \\
= \mathbb{E}\left[ \exp\left( \frac{h\| \alpha \|^2}{2} \frac{1}{h} \int_t^{t+h} \| X(u) \|^2 \, du \right) \right] \\
\leq \frac{1}{h} \int_t^{t+h} \mathbb{E}\left[ \exp\left( \frac{h\| \alpha \|^2}{2} \| X(u) \|^2 \right) \right] \, du \\
\leq \sup_{t \leq u \leq t+h} \mathbb{E}\left[ \exp\left( \frac{h\| \alpha \|^2}{2} \| X(u) \|^2 \right) \right].
\]

As \( X(u) \sim N(0, \Sigma_u) \), it follows that

\[
\mathbb{E}[\| X(u) \|^2] = \text{tr}(\Sigma_u) \leq \text{tr}(\Sigma).
\]

A “complete-a-square argument” in turn shows that, for all \( 0 < h < 1/\| \alpha \|^2 \text{tr}(\Sigma) \):

\[
\mathbb{E}\left[ \exp\left( \frac{h\| \alpha \|^2}{2} \| X(u) \|^2 \right) \right] = \frac{1}{1 - h\| \alpha \|^2 \text{tr}(\Sigma_u)} \leq \frac{1}{1 - h\| \alpha \|^2 \text{tr}(\Sigma)} < \infty.
\]

Therefore choosing \( h_0 = 1/\| \alpha \|^2 \text{tr}(\Sigma) \), (D.12) follows from

\[
\mathbb{E}\left[ \exp\left( \frac{1}{2} \int_t^{t+h} \| \alpha^\top X(u) \|^2 \, du \right) \right] \leq \sup_{t \leq u \leq t+h} \mathbb{E}\left[ \exp\left( \frac{h\| \alpha \|^2}{2} \| X(u) \|^2 \right) \right] \leq \frac{1}{1 - h\| \alpha \|^2 \text{tr}(\Sigma)} < \infty.
\]

Now we turn to the proof of (D.13). To this end, again using the submultiplicativity of the matrix operator norm and the convexity of the exponential function twice, we have

\[
\mathbb{E}\left[ \exp\left( \frac{1}{2} \int_{t_{n-1}}^{t_n} \int_0^{u_{n-1}} \| \alpha^\top X(u) \|^2 \, dudv \right) \right] \leq \mathbb{E}\left[ \exp\left( \frac{\| \alpha \|^2}{2n} \frac{1}{t_n - t_{n-1}} \int_{t_{n-1}}^{t_n} \int_0^{u_{n-1}} \| X(u) \|^2 \, dudv \right) \right] \\
\leq \frac{1}{t_n - t_{n-1}} \int_{t_{n-1}}^{t_n} \mathbb{E}\left[ \exp\left( \frac{\| \alpha \|^2}{2n} \int_0^{u_{n-1}} \| X(u) \|^2 \, du \right) \right] \, dv \\
\leq \mathbb{E}\left[ \exp\left( \frac{\| \alpha \|^2}{2n} \frac{ht_n}{t_n} \int_0^{u_{n-1}} \| X(u) \|^2 \, du \right) \right] \\
\leq \frac{1}{t_n} \int_0^{t_n} \mathbb{E}\left[ \exp\left( \frac{\| \alpha \|^2}{2n} \frac{ht_n}{2n} \| X(u) \|^2 \right) \right] \, du \\
\leq \sup_{0 \leq u \leq t_n} \mathbb{E}\left[ \exp\left( \frac{\| \alpha \|^2}{2n} \frac{ht_n}{2n} \| X(u) \|^2 \right) \right].
\]

Next, notice that

\[
t_n = \sum_{m=1}^n \frac{h}{m} < nh.
\]

Now choose \( h_1 = 1/\| \alpha \| \sqrt{\text{tr}(\Sigma)} \). Then, for \( 0 < h < h_1 \):

\[
\mathbb{E}\left[ \exp\left( \frac{\| \alpha \|^2}{2n} \frac{ht_n}{2n} \| X(u) \|^2 \right) \right] \leq \mathbb{E}\left[ \exp\left( \frac{\| \alpha \|^2}{2n} \frac{ht_n}{2n} \| X(u) \|^2 \right) \right] = \frac{1}{1 - h^2\| \alpha \|^2 \text{tr}(\Sigma_u)}.
\]
For $0 < h < h_1$, (D.13) then follows from
\[
\mathbb{E} \left[ \exp \left( \frac{1}{2} \int_{t_{n-1}}^{t_n} \int_{0}^{u} \| \alpha^\top X(u) \|^2 dv \right) \right] \leq \sup_{0 \leq u \leq t_n} \mathbb{E} \left[ \exp \left( \frac{\| \alpha \|^2 h t_n}{2 n} \| X(u) \|^2 \right) \right] \leq \sup_{0 \leq u \leq t_n} \frac{1}{1 - h^2 \| \alpha \|^2 \text{tr}(\Sigma_u)} \leq \frac{1}{1 - h^2 \| \alpha \|^2 \text{tr}(\Sigma)} < \infty.
\]
This completes the proof.

The next ingredient we need is an estimate for matrix exponentials:

**Lemma D.3.** Consider a symmetric positive semidefinite matrix $A \in \mathbb{R}^{d \times d}$ and let $a > 0$ be its smallest non-zero singular value if $\| A \| > 0$ and $a = 1$ when $\| A \| = 0$. Then, for all $t \geq 0$:
\[
\left\| A e^{-A^2 t} \right\| \leq \| A \| e^{-a^2 t}. \quad (D.14)
\]

**Proof.** When $\| A \| = 0$, the assertion evidently holds, so suppose $\| A \| > 0$. Since $A$ is symmetric positive semidefinite, there exists an orthonormal matrix $O$ such that
\[
A = O \text{ diag} \left( a_{(d_0)}, \ldots, a_{(1)}, 0, \ldots, 0 \right) O^\top,
\]
where $d_0 = \text{rank}(A) \leq d$, and
\[
0 < a = a_{(1)} \leq a_{(2)} \leq \cdots \leq a_{(d_0)} = \| A \|.
\]
By definition of the matrix exponential, we have
\[
e^{-A^2 t} = \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!} A^{2n};
\]
in particular, $A$ and $e^{-A^2 t}$ commute. Further, notice that
\[
O A e^{-A^2 t} O^\top = \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!} O A^{2n+1} O^\top
\]
\[
= \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!} \text{diag} \left( a_{(d_0)}^{2n+1}, \ldots, a_{(1)}^{2n+1}, 0, \ldots, 0 \right)
\]
\[
= \text{diag} \left( \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!} a_{(d_0)}^{2n+1}, \ldots, \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!} a_{(1)}^{2n+1}, 0, \ldots, 0 \right)
\]
\[
= \text{diag} \left( a_{(d_0)} e^{-a_{(d_0)}^2 t}, \ldots, a_{(1)} e^{-a_{(1)}^2 t}, 0, \ldots, 0 \right).
\]
As a consequence,
\[
\| A e^{-A^2 t} \| = \| O A e^{-A^2 t} O^\top \| = \| \text{diag} \left( a_{(d_0)} e^{-a_{(d_0)}^2 t}, \ldots, a_{(1)} e^{-a_{(1)}^2 t}, 0, \ldots, 0 \right) \|
\leq a_{(d_0)} e^{-a_{(d_0)}^2 t} = \| A \| e^{-a^2 t}
\]
as asserted. \qed
After these preparations, we can now show that our exponential local martingale is indeed a true martingale by verifying the conditions of (Karatzas and Shreve [1998] Chapter 3, Corollary 5.14):

**Corollary D.4.** The stochastic exponential $\mathcal{E}(M) = \exp(M_t - \frac{1}{2} \langle M \rangle_t)$ is a true martingale.

**Proof.** In order to apply the estimate from Lemma D.3, let $c$ be the smallest non-zero singular value of $\Lambda^{-1/2} \Pi_{\theta \theta} \Lambda^{-1/2}$ when $\|\Pi_{\theta \theta}\| > 0$, and $c = 1$ when $\|\Pi_{\theta \theta}\| = 0$. Plugging in Equation (3.1) for $\Pi_{\theta \theta}$ in the second step and then using Lemma D.3 we obtain the following bound, for every $t > 0$:

$$\gamma \int_0^t \left\| (\Pi_{\theta \theta} \sigma_x - \sigma_p)^\top \Lambda^{-1/2} e^{-\Lambda^{-1/2} \Pi_{\theta \theta} \Lambda^{-1/2} (t-u)} \right\|^2 \, du$$

$$= \gamma \int_0^t \left\| e^{-\Lambda^{-1/2} \Pi_{\theta \theta} \Lambda^{-1/2} (t-u)} (\Pi_{\theta \theta} \sigma_x - \sigma_p)^\top \Lambda^{-1/2} e^{-\Lambda^{-1/2} \Pi_{\theta \theta} \Lambda^{-1/2} (t-u)} \right\| \, du$$

$$= \int_0^t \left\| e^{-\Lambda^{-1/2} \Pi_{\theta \theta} \Lambda^{-1/2} (t-u)} \Lambda^{-1/2} (\Pi_{\theta \theta} \Lambda^{-1/2} + r \Pi_{\theta \theta}) \Lambda^{-1/2} e^{-\Lambda^{-1/2} \Pi_{\theta \theta} \Lambda^{-1/2} (t-u)} \right\| \, du$$

$$= \int_0^t \left\| (\Lambda^{-1/2} \Pi_{\theta \theta} \Lambda^{-1/2} + r I_m)^{1/2} (\Lambda^{-1/2} \Pi_{\theta \theta} \Lambda^{-1/2})^{1/2} e^{-\Lambda^{-1/2} \Pi_{\theta \theta} \Lambda^{-1/2} (t-u)} \right\|^2 \, du$$

$$\leq \int_0^t \left\| (\Lambda^{-1/2} \Pi_{\theta \theta} \Lambda^{-1/2} + r I_m)^{1/2} (\Lambda^{-1/2} \Pi_{\theta \theta} \Lambda^{-1/2})^{1/2} e^{-\Lambda^{-1/2} \Pi_{\theta \theta} \Lambda^{-1/2} (t-u)} \right\|^2 \, du$$

$$\leq \frac{1}{2c} \left\| \Lambda^{-1/2} \Pi_{\theta \theta} \Lambda^{-1/2} + r I_m \right\| \int_0^t \left\| (\Lambda^{-1/2} \Pi_{\theta \theta} \Lambda^{-1/2})^{1/2} e^{-2c(t-u)} \, du \right\|$$

Now for the application of Lemma D.2, notice that the multivariate Ornstein-Uhlenbeck process $X(t)$ satisfies

$$X(t) \sim N \left( e^{A_x t} X_0, \int_0^t e^{A_x (t-u)} \sigma_x \sigma_x^\top e^{A_x^\top (t-u)} \, du \right).$$

As $A_x$ is stable, there exists a symmetric positive semidefinite matrix $\Sigma$ such that

$$\int_0^t e^{A_x (t-u)} \sigma_x \sigma_x^\top e^{A_x^\top (t-u)} \, du \leq \Sigma.$$}

The process $\tilde{X}(t) = X(t) - e^{A_x t} X_0$ then is also Gaussian with

$$\tilde{X}(t) \sim N \left( 0, \int_0^t e^{A_x (t-u)} \sigma_x \sigma_x^\top e^{A_x^\top (t-u)} \, du \right).$$

Next, the definition (3.9) of the trading rate $\dot{\theta}(t)$ implies

$$d \left( e^{\Lambda^{-1/2} \Pi_{\theta \theta} \Lambda^{-1/2} \dot{\theta}(t)} \right) = e^{\Lambda^{-1/2} \Pi_{\theta \theta} \Lambda^{-1/2}} \left( \dot{\theta}(t) + \Lambda^{-1/2} \Pi_{\theta \theta} \dot{\theta}(t) \right) dt$$

$$= -e^{\Lambda^{-1/2} \Pi_{\theta \theta} \Lambda^{-1/2}} \Lambda^{-1/2} (\Pi_{\theta x} X(t) + \Pi_\theta) dt$$

$$= -e^{\Lambda^{-1/2} \Pi_{\theta \theta} \Lambda^{-1/2}} \Lambda^{-1/2} (\Pi_{\theta x} \tilde{X}(t) + \Pi_\theta e^{A_x t} X_0 + \Pi_\theta) dt.$$
Integrating both sides of the equation and multiplying with $\Lambda^{-1/2}e^{-\Lambda^{-1/2}\Pi_\theta\Lambda^{-1/2}t}$, we obtain

$$
\theta(t) = \Lambda^{-1/2}e^{-\Lambda^{-1/2}\Pi_\theta\Lambda^{-1/2}t}\Lambda^{1/2}\theta_0
- \int_0^t \Lambda^{-1/2}e^{-\Lambda^{-1/2}\Pi_\theta\Lambda^{-1/2}(t-u)}\Lambda^{-1/2}(\Pi_{\theta x}e^{A_xt}X_0 + \Pi_\theta)\, du
- \int_0^t \Lambda^{-1/2}e^{-\Lambda^{-1/2}\Pi_\theta\Lambda^{-1/2}(t-u)}\Lambda^{-1/2}\Pi_{\theta x}\bar{X}(u)\, du.
$$

This represents $\theta$ as an integral of the mean-zero Gaussian process $\bar{X}$, and in turn allows to apply the second part of Lemma D.2.

Next, we focus on the quadratic variation process

$$
d\langle M \rangle_t = \left\| (\Pi_{\theta x}\sigma_x - \sigma_p)^\top \theta(t) - \sigma_x^\top (\Pi_{xx}X(t) - \Pi_x) \right\|^2 dt.
$$

Minkowski’s inequality, Hölder’s inequality and (D.16) yield the following series of estimates:

$$
\begin{align*}
\frac{1}{3} \left\| (\Pi_{\theta x}\sigma_x - \sigma_p)^\top \theta(t) - \sigma_x^\top (\Pi_{xx}X(t) - \Pi_x) \right\|^2 \\
= \frac{1}{3} \left\| (\Pi_{\theta x}\sigma_x - \sigma_p)^\top \theta(t) - \sigma_x^\top (\Pi_{xx}\bar{X}(t) + \Pi_{xx}e^{A_xt}X_0 - \Pi_x) \right\|^2 \\
\leq C_t^2 + \left\| \sigma_x^\top \Pi_{xx}\bar{X}(t) \right\|^2 + \left\| (\Pi_{\theta x}\sigma_x - \sigma_p)^\top \int_0^t \Lambda^{-1/2}e^{-\Lambda^{-1/2}\Pi_\theta\Lambda^{-1/2}(t-u)}\Lambda^{-1/2}\Pi_{\theta x}\bar{X}(u)\, du \right\|^2 \\
\leq C_t^2 + \left\| \sigma_x^\top \Pi_{xx}\bar{X}(t) \right\|^2 + \left\| (\Pi_{\theta x}\sigma_x - \sigma_p)^\top \int_0^t \Lambda^{-1/2}e^{-\Lambda^{-1/2}\Pi_\theta\Lambda^{-1/2}(t-u)}\Lambda^{-1/2}\Pi_{\theta x}\bar{X}(u)\, du \right\|^2 \\
\leq C_t^2 + \left\| \sigma_x^\top \Pi_{xx}\bar{X}(t) \right\|^2 + \frac{1}{2\gamma c} \left\| \Lambda^{-1/2}\Pi_\theta\theta\Lambda^{-1/2} + rI_m \right\| \left\| \Lambda^{-1/2}\Pi_\theta\theta\Lambda^{-1/2} \right\| \left\| \Lambda^{-1/2}\Pi_{\theta x} \right\|^2 du \\
\left\| \Lambda^{-1/2}\Pi_\theta\theta\Lambda^{-1/2} + rI_m \right\| \left\| \Lambda^{-1/2}\Pi_\theta\theta\Lambda^{-1/2} \right\| \left\| \Lambda^{-1/2}\Pi_{\theta x} \right\|^2 du,
\end{align*}
$$

where

$$
C_t = (\Pi_{\theta x}\sigma_x - \sigma_p)^\top \Lambda^{-1/2}e^{-\Lambda^{-1/2}\Pi_\theta\theta\Lambda^{-1/2}t}\Lambda^{1/2}\theta_0 - \sigma_x^\top (\Pi_{xx}e^{A_xt}X_0 - \Pi_x)
- (\Pi_{\theta x}\sigma_x - \sigma_p)^\top \int_0^t e^{-\Lambda^{-1/2}\Pi_\theta\theta\Lambda^{-1/2}(t-u)}\Lambda^{-1/2}(\Pi_{\theta x}e^{A_xt}X_0 + \Pi_\theta)\, du.
$$

Now, choose

$$
0 < h < \frac{1}{6\sqrt{\text{tr}(\Sigma)}} \min \left\{ \frac{1}{\left\| \Pi_{xx}\sigma_x \right\|^2\sqrt{\text{tr}(\Sigma)}}, \frac{2\gamma c}{\left\| \Lambda^{-1/2}\Pi_\theta\theta\Lambda^{-1/2} + rI_m \right\| \left\| \Lambda^{-1/2}\Pi_\theta\theta\Lambda^{-1/2} \right\| \left\| \Lambda^{-1/2}\Pi_{\theta x} \right\|^2} \right\},
$$

and define $t_0 = 0$, $t_n = \sum_{m=1}^n \frac{h}{m}$. Then by (D.12), we have

$$
\mathbb{E} \left[ \exp \left( \int_{t_{n-1}}^{t_n} \left\| \sigma_x^\top \Pi_{xx}\bar{X}(t) \right\|^2 \, dt \right) \right] \leq \frac{1}{1 - 6h\left\| \Pi_{xx}\sigma_x \right\|^2\sqrt{\text{tr}(\Sigma)}} < \infty, \tag{D.17}
$$

and by (D.13),

$$
\mathbb{E} \left[ \exp \left( \frac{3}{2c} \left\| \Lambda^{-1/2}\Pi_\theta\theta\Lambda^{-1/2} + rI_m \right\| \left\| \Lambda^{-1/2}\Pi_\theta\theta\Lambda^{-1/2} \right\| \int_{t_{n-1}}^{t_n} \int_0^t \left\| \Lambda^{-1/2}\Pi_{\theta x}\bar{X}(u) \right\|^2 \, du \, dt \right) \right]
\leq \frac{1}{c} \frac{1}{1 - 3\left\| \Lambda^{-1/2}\Pi_\theta\theta\Lambda^{-1/2} + rI_m \right\| \left\| \Lambda^{-1/2}\Pi_\theta\theta\Lambda^{-1/2} \right\| \left\| \Lambda^{-1/2}\Pi_{\theta x} \right\|^2\text{tr}(\Sigma)} < \infty. \tag{D.18}
$$
Furthermore, notice that by H"older's inequality, for some large constant $K$,

$$
\mathbb{E} \left[ \exp \left( \frac{1}{2} \int_{t_{n-1}}^{t_n} d\langle M \rangle_t \right) \right] 
\leq e^{\frac{3}{2}\|C_t\|^2} \mathbb{E} \left[ \exp \left( 3 \int_{t_{n-1}}^{t_n} \|\sigma_x^\top \tilde{X}(t)\|^2 dt \right) \right] 
\leq e^{\frac{3}{2}\|C_t\|^2} \mathbb{E} \left[ \exp \left( K \int_{t_{n-1}}^{t_n} \int_0^t \|\Lambda^{-1/2} \Pi_{\theta x} \tilde{X}(u)\|^2 du dt \right) \right]^{1/2} 
< \infty.
$$

Finally, since $\sum_{n=1}^{\infty} \frac{h}{m_n} = \infty$, we have $t_n \to \infty$ as $n \to \infty$. The asserted true martingale property therefore follows from \parencite{Karatzas and Shreve 1998} Chapter 3, Corollary 5.14).

\section*{E Proofs for Section 3.2}

\subsection*{E.1 Relation between the Pricing Kernel and Trade Price}

\vspace{0.5cm}

It\'o's formula and the dynamics \parencite{3.11} of the pricing kernel and \parencite{3.14} of the trade price yield

\begin{align*}
d \left( \phi(t) \tilde{P}(t) \right) 
&= \phi(t) d\tilde{P}(t) + \tilde{P}(t) d\phi(t) + d\langle \phi \tilde{P} \rangle(t) \\
&= \phi(t) \left( \mu + \hat{C}_x X(t) + r \phi(t) \tilde{P}(t) - r \phi(t) \tilde{P}(t) - \gamma \tilde{\sigma}_p \sigma_x^\top (\Pi_{xx} X(t) - \Pi_x) - \gamma \tilde{\sigma}_p \sigma_x^\top \theta^\top (X(t)) \right) dt \\
&\quad - \gamma \phi(t) \left( \tilde{P}(t) \left( \theta(t)^\top \tilde{\sigma}_p + (\Pi_{xx} X(t) - \Pi_x)^\top \sigma_x \right) + \tilde{\sigma}_p \right) dB(t) \\
&= \phi(t) \left( \mu + \gamma (\Pi_{\theta \theta} \Lambda^{-1} + r I_m) \Pi_{\theta} + \gamma \tilde{\sigma}_p \sigma_x^\top \Pi_x + \left( \tilde{C}_x + \gamma (\Pi_{\theta \theta} \Lambda^{-1} + r I_m) \Pi_{\theta x} - \gamma \tilde{\sigma}_p \sigma_x^\top \Pi_{xx} \right) X(t) \right) dt \\
&\quad - \gamma \phi(t) \left( \tilde{P}(t) \left( \theta(t)^\top \tilde{\sigma}_p + (\Pi_{xx} X(t) - \Pi_x)^\top \sigma_x \right) + \tilde{\sigma}_p \right) dB(t) \\
&= -\gamma \phi(t) \left( \tilde{P}(t) \left( \theta(t)^\top \tilde{\sigma}_p + (\Pi_{xx} X(t) - \Pi_x)^\top \sigma_x \right) + \tilde{\sigma}_p \right) dB(t) \\
&= -\gamma \phi(t) \left( \tilde{P}(t) \left( \theta(t)^\top \tilde{\sigma}_p + (\Pi_{xx} X(t) - \Pi_x)^\top \sigma_x \right) + \tilde{\sigma}_p \right) dB(t).
\end{align*}

Here, $\mu + \gamma (\Pi_{\theta \theta} \Lambda^{-1} + r I_m) \Pi_{\theta} + \gamma \tilde{\sigma}_p \sigma_x^\top \Pi_x = 0$ follows from \parencite{3.4} and $\tilde{C}_x + \gamma (\Pi_{\theta \theta} \Lambda^{-1} + r I_m) \Pi_{\theta x} - \gamma \tilde{\sigma}_p \sigma_x^\top \Pi_{xx}$ is a consequence of \parencite{3.3}. The martingale property of the local martingale $\phi(t) \tilde{P}(t)$ follows from Lemma \parencite{D.2}.

\subsection*{E.2 Sufficient Condition for Invertibility of $\Pi_{\theta \theta}$}

We now provide a sufficient condition on the primitives of the model that ensures the invertibility of the matrix $\Pi_{\theta \theta}$, which allows to rewrite the optimal trading rate in Theorem \parencite{3.1} as trading towards a target portfolio.

\textbf{Proposition E.1.} $\Pi_{\theta \theta}$ is invertible if $\text{rank} \left( \sigma_x^\top \sigma_x + C_x (r I_d - A_x)^{-1} \sigma_x \sigma_x^\top \right) = m$. 

Electronic copy available at: https://ssrn.com/abstract=4522752
Proof. Suppose rank \((\sigma_p\sigma_x^T + C_x(rI_d - A_x)^{-1}\sigma_x\sigma_x^T)\) = \(m\). We establish the invertibility of \(\Pi_{\theta\theta}\) by contradiction. Suppose there exists \(\theta_0 \in \mathbb{R}^m\) with \(\|\theta_0\| = 1\) such that \(\Pi_{\theta\theta}\theta_0 = 0\). Multiplying by \(\theta_0\) on both sides of (3.3) from the left, we obtain

\[
\gamma (\Pi_{\theta x}\sigma_x - \sigma_p) (\Pi_{\theta x}\sigma_x - \sigma_p)^T \theta_0 = \Pi_{\theta\theta} \Lambda^{-1} \Pi_{\theta\theta} \theta_0 + r \Pi_{\theta\theta} \theta_0 = 0.
\]

Whence, 0 is an eigenvalue of \((\Pi_{\theta x}\sigma_x - \sigma_p) (\Pi_{\theta x}\sigma_x - \sigma_p)^T\). As a consequence,

\[
\theta_0^T \Pi_{\theta x} \sigma_x = \theta_0^T \sigma_p
\]

and in turn

\[
\theta_0^T \Pi_{\theta x} = \theta_0^T \sigma_p \sigma_x^T \left(\sigma_x\sigma_x^T\right)^{-1}.
\]

Therefore, after multiplying both sides of (3.3) with \(\theta_0^T\), we obtain

\[
0 = \theta_0^T \left(\gamma \sigma_p \sigma_x^T \Pi_{xx} - C_x - \left(\Pi_{\theta\theta} \Lambda^{-1} + \frac{r}{2} I_m\right) \Pi_{\theta x} - \Pi_{\theta x} \left(\frac{r}{2} I_d - A_x + \gamma \sigma_x \sigma_x^T \Pi_{xx}\right)\right)
\]

\[
= \gamma \theta_0^T \sigma_p \sigma_x^T \Pi_{xx} - \theta_0^T C_x - \theta_0^T \Pi_{\theta\theta} \Lambda^{-1} \Pi_{\theta x} - \theta_0^T \Pi_{\theta x} \left(\frac{r}{2} I_d - A_x + \gamma \sigma_x \sigma_x^T \Pi_{xx}\right)
\]

\[
= \gamma \theta_0^T \sigma_p \sigma_x^T \Pi_{xx} - \theta_0^T C_x - \theta_0^T \sigma_p \sigma_x^T \left(\sigma_x\sigma_x^T\right)^{-1} \left(\frac{r}{2} I_d - A_x + \gamma \sigma_x \sigma_x^T \Pi_{xx}\right)
\]

\[
= -\theta_0^T \left(C_x + \sigma_p \sigma_x^T \left(\sigma_x\sigma_x^T\right)^{-1} \left(\frac{r}{2} I_d - A_x\right)\right)
\]

\[
= -\theta_0^T \left(\sigma_p + C_x (rI_d - A_x)^{-1}\sigma_x\right) \sigma_x^T \left(\sigma_x\sigma_x^T\right)^{-1} (rI_d - A_x).
\]

Since \(\left(\sigma_x\sigma_x^T\right)^{-1} (rI_d - A_x)\) is invertible,

\[
\theta_0^T \left(\sigma_p + C_x (rI_d - A_x)^{-1}\sigma_x\right) \sigma_x^T = 0.
\]

As \(\left(\sigma_p + C_x (rI_d - A_x)^{-1}\sigma_x\right)\) has full rank, we infer that \(\theta_0 = 0\), which contradicts that \(\|\theta_0\| = 1\). Hence \(\Pi_{\theta\theta}\) is invertible.

\[
\square
\]

References
