Relationship between deep hedging and delta hedging: leveraging a statistical arbitrage strategy

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Abstract
Recently, a new approach, deep hedging has been developed to identify optimal hedging strategies in incomplete markets, to minimize potential losses. However, the relationship between risk-neutral delta hedging and deep hedging is complicated. For deep hedging to be useful in practice, we must align with established risk-neutral delta hedging methods. In this study, we explore the relationship between deep and delta hedging using a statistical arbitrage strategy. Specifically, we show that hedging that minimizes loss risk combines delta hedging with a statistical arbitrage strategy. We also discuss how statistical arbitrages can hamper deep hedging. Finally, we analyze the profit and loss distribution of deep hedging and discuss the risk measures that resist statistical arbitrage.

Keywords: Deep hedging, Risk neutral delta hedging, Risk measures

1. Introduction

Pricing and delta hedging based on risk-neutral valuation in complete markets are the prevailing approaches (Hull 2003). When hedging based on a risk-neutral valuation, it is essential to first calculate the price of the derivative
being hedged. Specifically, the cost of replication is the price, guaranteed by the replication/no-arbitrage principle (Harrison & Kreps 1979; Harrison & Pliska 1983).

In a complete market, where there are no transaction costs, market frictions, or differences in lending or borrowing interest rates, and one can trade any quantity at any time, such hedging is entirely feasible (Ku et al. 2012; Hirsa & Neftci 2013). However, markets are not always complete. In reality, achieving a perfect hedge is a rare occurrence (Hull 2003).

Incomplete markets are situations that stray from an idealized complete market. To address the challenges of pricing and hedging in incomplete markets, a deep learning-based framework called ‘deep hedging’, has emerged in recent literature (Buehler et al. 2019). This approach prioritizes minimizing the loss risk in such markets (Xu 2006). The versatility of deep hedging is evident in its successful application to ultra-long-term derivatives (Carbonneau 2021), its adaptability to LSTM network structures (Zhang & Huang 2021), and the integration of an inductive bias termed ‘No transaction band’ (Imaki et al. 2023), which enhances hedging efficiency. Furthermore, there is an evolving discourse on shifting the focus from merely minimizing loss risk to adopting equal risk pricing (Carbonneau & Godin 2021).

However, the relationship between delta hedging based on risk-neutral valuations and deep hedging based on minimizing loss risk has not been fully investigated. For deep hedging to function effectively in financial practice, we must compare it with existing risk-neutral evaluations and ensure compatibility. Although the setting in Buehler et al. (2019) highlights the congruence of the two approaches, their general links remain a subject of inquiry.

Additionally, deep hedging can be skewed by statistical arbitrage opportunities arising from a positive drift in the price process of hedging instruments. Specifically, the model may prioritize profits from trading these assets instead of focusing on the payoffs from contingent claims. Buehler et al. (2021) tackled this problem by adjusting the risk-neutral probability measure, which helped create a more accurate hedge, particularly for option hedging. However, a clear
definition and understanding of statistical arbitrage opportunities is still needed to address this issue more directly.

Therefore, in this study, we explore the relationship between deep and delta hedging using statistical arbitrages. We begin by defining the statistical arbitrage strategies. We then explore the theoretical connection between deep hedging, which aims to minimize loss risk, and delta hedging for replicable contingent claims. Specifically, we show that deep hedging combines delta hedging and statistical arbitrage. Using this representation, we discuss that Deep hedging can be hampered by the existence of statistical arbitrages. Finally, we analyze the profit and loss (PnL) distribution of deep hedging and discuss the risk measures that resist statistical arbitrages.

2. Delta Hedging and Deep Hedging

2.1. Market and Hedging Strategies

We assume a continuous-time setting with finite maturity $T > 0$. Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$, with $\mathbb{P}$ being the real-world probability measure and $\mathbb{F} := \{\mathcal{F}_t\}_{t \in [0,T]}$ being a filtration satisfying the usual condition. $S := \{S_t\}_{t \in [0,T]}$ is a one-dimensional non-negative right continuous with left limits (RCLL) adapted process representing the price process of the underlying asset. We price and hedge a contingent claim with payoff $Z$ at time $T$, which is $\mathcal{F}_T$-measurable. Following Buehler et al. (2019), for simplicity, we assume that all intermediate payments accumulate at the risk-free interest rate. We now define $\mathcal{H} := \left\{ \delta = \{\delta_t\}_{t \in [0,T]} \mid \delta : \text{admissible strategy} \right\}$. We choose a hedging strategy from this set. Throughout the study, we assume no transaction costs in the market and that the market is arbitrage-free.

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1The main results hold even in multi-dimensional settings.
2That is, we assume that all payments are made at time $T$. We exclude contingent claims such as American options.
3Strategies are considered equivalent when they are indistinguishable.
Assumption 1. Z is replicable, meaning there exists an admissible delta hedging strategy \( \Delta \in \mathcal{H} \), such that \( Z = p_0 + (\Delta \cdot S)_T \). Here, \( p_0 \) is the arbitrage-free price or the risk-neutral price, and \( (\delta \cdot S)_T := \int_0^T \delta_t dS_t \) represents the cumulative gains and losses arising from hedging assets (underlying assets) for \( \delta \in \mathcal{H} \).

2.2. Risk Measure, Deviation, and Sharpe ratio

Buehler et al. (2019) propose the hedging method using a convex risk measure as an optimality criterion.

Definition 1 (Convex Risk measure). Let \( \chi \) be a set of random variables representing asset positions and \( X, X_1, X_2 \in \chi \). We call \( \rho : \chi \to \mathbb{R} \) a convex risk measure if it satisfies the following conditions.

- **(R1) Monotone decreasing:** if \( X_1 \leq X_2 \) then \( \rho(X_1) \geq \rho(X_2) \) for \( \alpha \in [0, 1] \).
  
  In simple terms, having larger asset positions should reduce risk.

- **(R2) Convex:** \( \rho(\alpha X_1 + (1 - \alpha) X_2) \leq \alpha \rho(X_1) + (1 - \alpha) \rho(X_2) \).
  
  This implies that diversification should not increase risk.

- **(R3) Cash-invariant:** \( \rho(X + c) = \rho(X) - c \) for \( c \in \mathbb{R} \).
  
  This means that the hedger does not differentiate a zero portfolio from a portfolio with asset \( X \) and cash \( \rho(X) \), because \( \rho(X + \rho(X)) = \rho(X) - \rho(X) = 0 \).

Example 1 (Entropic Risk Measure). \( \rho(X) := \frac{1}{\theta} \log \mathbb{E}_p [e^{-\theta X}] \), for \( \theta > 0 \) is an entropic risk measure and is an important example of a convex risk measure.

When we strengthen the assumptions of a risk measure, we obtain a Coherent Risk measure.

Definition 2 (Coherent Risk Measure). A convex risk measure, denoted by \( \rho \), is called a Coherent Risk Measure if it satisfies the additional condition:

In simple terms, this refers to a trading strategy that does not take infinite risk.
(R4) Positive homogeneity: For \( \lambda \geq 0 \), \( \rho(\lambda X) = \lambda \rho(X) \)

If we adjust the position by multiplying it by a positive constant, then the corresponding risk increases proportionally.

To analyze the link between risk and deviation measures (see Theorem 1), we require the following type of risk measure:

**Definition 3** (Strictly expectation bounded risk measure). Suppose \( \rho : \chi \to \mathbb{R} \) satisfies (R2), (R3) and (R4), along with

(R5) \( \mathcal{R}(X) > -\mathbb{E}[X] \)

We call \( \rho \) a strictly expectation bounded risk measure.

**Example 2.** CVaR\(_{\alpha}(X) := \frac{1}{\alpha} \int_0^\alpha \text{VaR}_\lambda(X)d\lambda, \) for \( \theta > 0 \) is Conditional Value at Risk (CVaR) which evaluates the tail risk of loss. CVaR is an example of both coherent risk measures and strictly expectation bounded risk measures.

Subsequently, Rockafellar et al. (2006) introduced the concept of deviation as follows.

**Definition 4** (Deviation Measure). Let \( \chi \) be a \( \mathcal{L}^2 \) space of random variables representing discounted asset positions and \( X, X_1, X_2 \in \chi \). We call a functional \( \mathcal{D} : \chi \to \mathbb{R} \) a deviation measure if it satisfies (R1) - (R4).

(D1) \( \mathcal{D}(X + c) = \mathcal{D}(X) \) for all \( X \) and \( c \in \mathbb{R} \).

(D2) Positive homogeneity: \( \mathcal{D}(0) = 0 \) and \( \mathcal{D}(\lambda X) = \lambda \mathcal{D}(X) \) for all \( X \) and all \( \lambda > 0 \)

(D3) Sub-additivity: \( \mathcal{D}(X_1 + X_2) \leq \mathcal{D}(X_1) + \mathcal{D}(X_2) \) for \( \alpha \in \mathbb{R} \).

(D4) Positivity: \( \mathcal{D}(X) \geq 0 \) for all \( X \), with \( \mathcal{D} > 0 \) for nonconstant \( X \).

From (D1), we observe that the deviation vanishes only when \( X - \mathbb{E}[X] = 0 \). Thus, as noted in Rockafellar et al. (2006), deviation measures capture the uncertainty in asset position \( X \). Standard deviation (volatility) is an example of a deviation measure.
Additionally, Rockafellar et al. (2006) shows a one-to-one relationship between a deviation measure and a strictly expectation bounded risk measure; the study obtains the following result.

**Theorem 1 (Rockafellar et al. (2006)).** Deviation measures correspond one-to-one with a strictly expectation bounded under the relations

\[ \rho(X) = \mathcal{D}(X) - \mathbb{E}[X], \quad \mathcal{D}(X) = \rho(X - \mathbb{E}[X]) \]

2.3. Risk Indifferent Pricing and Hedging

Buehler et al. (2019) defines the indifference price of a contingent claim, using a value of a convex risk measure as a criterion of optimal hedging.

First, we consider the following optimization problem:

\[ \pi(X) := \inf_{\delta \in \mathcal{H}} \rho(X + (\delta \cdot S)_T) \]

Note that we ignore the term for accumulated transaction costs because we assume that the market is free of friction, unlike Buehler et al. (2019). Therefore, the PnL of the hedging strategy is determined purely from the liability and price changes in \( S_t \). Subsequently, the optimal hedging problem for future liability \(-Z\) at time \( T\) is defined as follows.

\[ \pi(-Z) = \inf_{\delta \in \mathcal{H}} \rho(-Z + (\delta \cdot S)_T) \]

According to Buehler et al. (2019), we can interpret \( \pi(-Z) \) as the minimum amount of cash needed to be invested at the initial time to make the possible liability acceptable with regard to the risk measure. Now, we define the price of the contingent claim \( p(Z) \) as

\[ p(Z) := \pi(-Z) - \pi(0) \]

Here, \( \pi(0) \) should be considered in the case that having no liability (taking no risks) has a positive value in terms of the risk measure of the hedger. Therefore, we call this price the indifference price \( p(Z) \) because this is the amount of cash necessary for the hedger to not differentiate liability \(-Z\) from zero liability.
Buehler et al. (2019) shows that in the complete market, this risk-indifferent price is equal to the Risk-Neutral price.

**Proposition 1** (Buehler et al. (2019)). \( p(Z) = p_0 \)

*Proof.* The proof is provided in the Appendix.

2.4. Relationship between Delta hedging and Deep hedging

Here, we discuss the relationship between delta and deep hedging. The deviation between those strategies can be explained by the statistical arbitrage strategy under the risk measure.

**Definition 5** (Statistical Arbitrage Strategy). If \( \delta \in \mathcal{H} \) satisfies the following condition, then we call \( \delta \) **Statistical Arbitrage Opportunity**:

\[
\rho((\delta \cdot S)_T) < 0
\]

In addition, a statistical arbitrage opportunity \( \delta^{SA} \) satisfies the following, we call \( \delta \) **Statistical Arbitrage Strategy**:

\[
\rho((\delta^{SA} \cdot S)_T) = \pi(0) < 0
\]

**Theorem 2** (Relationship between Delta Hedging and Deep Hedging).

*Assume that there exists a unique statistical arbitrage strategy \( \delta^{SA} \) and that \( \delta^{DH} \in \mathcal{H} \) satisfies:

\[
\rho(-Z + (\delta^{DH} \cdot S)_T) = \pi(-Z)
\]

Then, it follows

\[
\delta^{DH} = \delta^{SA} + \Delta
\]

*Proof.* The proof is provided in the Appendix.

**Remark 1.** This proposition shows that the gap between delta hedging and deep hedging is derived from the optimal strategy that minimizes the value of the risk measure by the gain from underlying trades.

Furthermore, this additional assumption yields the following result:
Proposition 2 (A condition for delta hedging and deep hedging to coincide). Suppose there exists no statistical arbitrage opportunity i.e. \( \pi(0) \geq 0 \). Then,

\[ \rho(-Z + (\Delta \cdot S)_T) = \pi(-Z) \]

*Proof.* The proof is provided in the Appendix.

Proposition 3 (Another condition with a strictly expectation bounded risk measure for delta hedging and deep hedging to coincide). Suppose \( \rho \) is a strictly expectation bounded risk measure, \( S \) is a continuous, square-integrable \( \mathbb{P} \)-martingale and a set of possible strategy is limited to \( \mathcal{H}^2 := \{ \delta = (\delta_t)_{t \in [0,T]} | \text{predictable}, \mathbb{E}[\int_0^T |\delta_t|^2 d[S]_t] < \infty \} \) Then, no statistical arbitrage opportunity exists in \( \mathcal{H}^2 \)

*Proof.* The proof is provided in the Appendix.

3. Numerical Experiment

In this section, we analyze how deep hedging and delta hedging diverge from each other through a numerical experiment and verify the facts proven in Chapter 2.3.

As the underlying, we adopt geometric Brownian motion:

\[ dS_t = \mu S_t dt + \sigma S_t dW_t, \quad \mu > 0, \sigma > 0. \]

We try to hedge an At The Money (ATM) European call option.

We use the entropic risk measure defined below for the risk measure:

\[ \text{ENT}_\theta(X) = \frac{1}{\theta} \log \mathbb{E}_\mathbb{P} \left[ e^{-\theta X} \right], \quad \theta > 0 \]

By solving the appropriate stochastic control problems (see Appendix B), the statistical arbitrage strategy is obtained as follows:

\[ \delta^{SA} = \left\{ \frac{\mu}{\theta \sigma^2 S_t} \right\}_{t \in [0,T]} \]
From Theorem 2, the optimal deep hedging strategy is:

$$\delta^{DH} = \left\{ \Delta_t + \frac{\mu}{\theta \sigma^2 S_t} \right\}_{t \in [0, T]} ,$$

where $\Delta$ is Black-Scholes's delta.

We consider a discrete-time market in which $T = 0.08$, $\Delta t = 0.004$. We use a feed-forward neural network with four hidden layers with 32 neurons. For the activation function, we adopt ReLU. We use Adam with the default parameters for optimization. To stabilize the learning process, the initial value of the underlying is set to 1.

The upper boxes in Figure 1 represent $|\delta^{DH} - \Delta|/|\Delta|$, that is, the standardized difference between the delta hedging strategy and learned deep hedging strategy in this setting. The lower boxes denote $|\delta^{DH} - \Delta - \delta^{SA}|/|\Delta|$, that is, the difference between the left and right-hand side of the equation in Theorem 2.

We observed the deviations at different levels of drift and volatility. Note that the gap between deep hedging and delta hedging increases linearly as the drift of the underlying asset goes up (1a). Also, when we increase the volatility term, the hedging error decreases nonlinearly (1b). Interestingly, the adjusted value, which represents the difference in hedging minus the statistical arbitrage strategy, remains close to zero, irrespective of the levels of drift and volatility. This agrees well with the claims of Theorem 2.

We also note that the statistical arbitrage strategy has a nonlinear relationship with absolute stock price levels. Specifically, it is inversely proportional to absolute stock price levels. When the stock price is high, delta hedging and deep hedging are approximately equal.

4. Discussion

4.1. Comparison between Entropic Risk Measure and CVaR

We now discuss the performance and robustness of the deep hedging strategies learned with entropic risk measure, and conditional value at risk in presence of statistical arbitrage opportunities.
Figure 1: The upper boxes represent the spread of $\frac{||\delta^{DH} - \Delta||}{||\Delta||}$ calculated for 1000 simulations while lower ones do $\frac{||\delta^{DH} - \Delta - \delta^{SA}||}{||\Delta||}$. $\delta^{DH}$ is trained with ENT$_{1.0}$. The level of $\frac{||\delta^{DH} - \Delta||}{||\Delta||}$ is relatively larger for greater drifts and smaller volatilities. Conversely, that of $\frac{||\delta^{DH} - \Delta - \delta^{SA}||}{||\Delta||}$ remains around 0 regardless of the values of parameters based on the observations of the lower boxes.

![Figure 1](image)

(a) Volatility is fixed at $\sigma = 0.20$.  
(b) Drift is fixed at $\mu = 0.05$.

Figure 2 shows the spread of $\frac{||\delta^{DH} - \Delta||}{||\Delta||}$ with $\delta^{DH}$ learned using CVaR$_{0.5}$ for different $\mu$ and $\sigma$. Figure 3 shows the distribution of PnL for the different strategies for 100,000 simulations. The parameters were set as $\mu = 0.05$, $\sigma = 0.20$.

As Figure 1a, 1b when we use ENT$_{1.0}$, deep hedging is significantly influenced by the presence of a statistical arbitrage strategy.

Figure 3a shows that a statistical arbitrage strategy can hinder the learning of hedging, and greatly inflate the tail of the PnL distribution. Conversely, when we use CVaR$_{0.5}$, the difference remains constant relative to the drift term level, as shown in Figure 2a. Additionally, for the different volatility terms, the differences are smaller than those of ENT$_{1.0}$. We can also confirm in Figure 3b that its PnL approximates the performance of delta hedging, meaning that the proper hedging strategy has been learned.

Robustness against the impact of statistical arbitrage opportunities depends on the extent to which each risk measure is influenced by the profits from trading the hedging instruments. As shown in Figure 3a, ENT$_{1.0}$ is significantly affected
Figure 2: Each box represents the spread of $\frac{|\delta^{DH} - \Delta|}{|\Delta|}$ calculated for 1000 simulations. $\delta^{DH}$ is trained with CVaR$_{0.5}$. They almost maintain a level near 0 with different drifts and volatilities.

by the profits made through the trade of the underlying asset. However, CVaR$_{0.5}$ is influenced only by the left half (tail risk) of the distribution, resulting in subtle sensitivity to gains from trading. Therefore, CVaR$_{0.5}$ is considered robust.

It is important to highlight that ENT$_{\theta}$ can be used as a robust risk measure depending on the parameter selection. Figure 4 shows the robustness of the deep hedging strategies with ENT$_{100.0}$ against the existence of statistical arbitrage opportunities. Notably, increasing the coefficient parameter $\theta$ diminishes the risk value’s sensitivity to the gains from trading, as illustrated in Figure 4a and 4b. This relationship becomes evident when observing that the statistical arbitrage strategy is inversely related to constant $\theta$.

4.2. Advantages and caveats of robust risk measures

First, using robust risk measures allows us to ignore the estimation of the drift term in the asset pricing process.

Deep hedging, which employs optimization under a real probability measure $P$, requires a more rigorous incorporation of drift into the pricing model. However, it is difficult to estimate the drift term in practice.

As confirmed with CVaR$_{0.5}$, for a risk measure that is robust to the statistical arbitrage strategy arising from the drift term, the drift term levels in the price
Figure 3: Distributions of PnL for $\delta^{DH}$ with different risk measures. We set parameters as $\mu = 0.05$, $\sigma = 0.20$. Deep hedging strategy with CVaR$_{0.5}$ performs closely to the BS delta hedging strategy, while that with ENT$_{1.0}$ is significantly affected by the existence of statistical arbitrage opportunities.

(a) ENT$_{1.0}$

(b) CVaR$_{0.5}$

process do not make a substantial difference to the learned strategy. Therefore, the choice of such a risk measure justifies discarding the drift term estimation, even under a real-world probability measure.

However, care must be taken with pricing. When learning deep hedges with different risk measures, the derived hedging strategies approximate each other if they are robust to the statistical arbitrage strategy. However, differences naturally arise with respect to risk-neutral prices, and in this case the prices are difficult to interpret. Therefore, realistically, it is more appropriate to use other pricing approaches, such as risk-neutral pricing valuation methods. The charged costs should then be added to the final PnL, and optimization should be performed based on their distribution.

5. Conclusion

In this study, we first mathematically define a statistical arbitrage strategy that prevents clean learning in deep hedging. Next, we demonstrate that hedging based on loss risk minimization can be expressed as the total of delta hedging and statistical arbitrage strategies. Using this description, we discuss the features of the PnL distribution in deep hedging and a robust risk measure.
The increased robustness of deep hedging strategies with ENT_{100,0} against the existence of statistical arbitrage opportunities. The level of \( \frac{||\delta_{DH} - \Delta||}{||\Delta||} \) remains closely to 0 \( \text{(b)} \) and the performance as PnL also gets closer to that of BS delta hedging strategy than that of ENT_{1,0} \( \text{(a)} \).

for the statistical arbitrage strategy. With this risk measure, we can ignore the drift term in the pricing model, even under the real probability measure. Although we do not consider transaction costs, we believe that these robust risk measures will allow for clean learning even when transaction costs are more broadly present.

Future research should focus on the following topics, first, on the general properties of the hedging strategy \( \delta_{DH} - \delta_{SA} \), which is defined without relying on the replicability of conditional claims. Next, we investigate the robustness of CVaR_\alpha from a theoretical perspective. This can be accomplished by obtaining \( \delta_{SA} \) for CVaR_\phi analytically.

References


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Appendix A. Proofs

Appendix B. Deriving the Statistical Arbitrage Strategy for Entropic Risk Measure

Appendix A. Proofs

Proof of Proposition 1. Let $\delta \in \mathcal{H}$ be an arbitrary. Since the market is complete and there is the cash-invariance of $\rho$, it follows that

$$\rho (-Z + (\delta \cdot S)_T) = p_0 + \rho ([\delta - \Delta] \cdot S)_T$$

By taking the infimum for $\delta$ on both sides, we obtain

$$\pi(-Z) = p_0 + \pi(0)$$

Proof of Theorem 2. Proposition

$$\pi(-Z) - \pi(0) = p_0$$

$$\iff \rho (-Z + (\delta^{DH} \cdot S)_T) = p_0 + \rho ([\delta^{SA} \cdot S)_T$$

$$\iff \rho ((\delta^{DH} - \Delta) \cdot S)_T) = \rho ([\delta^{SA} \cdot S)_T$$

Then, based on the uniqueness of $\delta^{SA}$, we obtain the result.
Proof of Proposition 2. Let $\delta \in \mathcal{H}$ be an arbitrary. Here, since $\pi(0) \geq 0$, we obtain $\forall \delta \in \mathcal{H}, \rho((\delta \cdot S)_T) \geq 0$ Therefore, the non-negativity of the risk measure provides

\[
\rho(-Z + (\delta \cdot S)_T) = p_0 + \rho([\delta - \Delta] \cdot S)_T \geq p_0
\]

\[
= \rho(-Z + (\Delta \cdot S)_T)
\]

Therefore, $\Delta$ is the optimal hedging strategy in $\mathcal{H}$. In other words, $\Delta = \delta^{DH}$, if $\delta^{DH}$ is unique. \hfill $\Box$

Proof of Proposition 3. Since $S$ is a continuous square integrable $\mathbb{P}$-martingale, the stochastic integration $(\delta \cdot S)$ for $\delta \in \mathcal{H}^2$ is also a $\mathbb{P}$-martingale. As $\rho$ is a strictly expectation bounded risk measure, we can use Theorem[1] Therefore, for any $\delta \in \mathcal{H}^2$, we obtain

\[
\rho((\delta \cdot S)_T) = \mathcal{D}((\delta \cdot S)_T) - \mathbb{E}[(\delta \cdot S)_T] = \mathcal{D}((\delta \cdot S)_T) \geq 0
\]

This implies that $\pi(0) \geq 0$. Therefore, from Proposition[2] we obtain the result. \hfill $\Box$

Appendix B. Deriving the Statistical Arbitrage Strategy for Entropic Risk Measure

Consider the following problem to obtain the statistical arbitrage strategy.

\[
\inf_{\delta \in \mathcal{H}} \rho((\delta \cdot S)_T) \iff \inf_{\delta \in \mathcal{H}} \mathbb{E}_\mathbb{P}\left[e^{-\theta(\delta \cdot S)_T}\right]
\]

First, we consider the following Markov control problem.

\[
\inf_{\nu_t} \mathbb{E}_\mathbb{P}\left[e^{-\theta X_T} | \mathcal{F}_t\right] = \inf_{\nu_t} \mathbb{E}_\mathbb{P}\left[e^{-\theta X_T}| \mathcal{F}_t\right] \]

\[
dX_t = \nu_t \mu X_t dt + \nu_t \sigma X_t dW_t, \quad X_0 = S_0
\]
where $X_t$ is the portfolio value at time $t$ and $\nu_t$ is the weight of the underlying assets relative to the portfolio. From the Markov property of $X$, we define the value function:

$$V(t, X_t) := \inf_{\nu_t} \mathbb{E}_\nu [e^{-\theta X_T} | \mathcal{F}_t]$$

By using Bellman’s optimality principle, $V(t, X_t)$ satisfies

$$0 = \inf_{\nu_t} \frac{\partial V}{\partial t}(t, X_t) + \frac{\partial V}{\partial x}(t, X_t)\nu_t\mu X_t + \frac{1}{2} \frac{\partial^2 V}{\partial x^2}(t, X_t)\nu_t^2 \sigma^2 X_t^2 \quad (A-1)$$

Using the first order condition, we obtain

$$\nu_t^* = -\frac{\partial_x V \mu}{\partial_{xx} V \sigma^2 X_t}$$

We substitute this into Equation (A-1)

$$0 = \partial_t V - \frac{1}{2} \left( \frac{\partial_x V}{\partial x} \right)^2 \mu + \frac{\partial^2 V}{\partial x^2} V \sigma^2 x$$

The solution is:

$$V(t, x) = e^{-\theta x - \frac{1}{2} \frac{\mu}{\sigma^2} (T-t)} \quad (A-2)$$

As $\frac{\partial V}{\partial t} = \frac{1}{2} \frac{\mu}{\sigma^2} V(t, x), \frac{\partial V}{\partial x} = -\theta V(t, x), \frac{\partial^2 V}{\partial x^2} = \theta^2 V(t, x)$, (A-2) is the solution for the PDE (A-1). Thus, the optimal control $\nu_t^*$ is,

$$\nu_t^* = \frac{\mu}{\theta \sigma^2 X_t}$$

This optimal control and statistical arbitrage strategy satisfies the following equation:

$$\delta_t^{SA} = \nu_t^* X_t$$

Therefore, we obtain the following statistical arbitrage strategy:

$$\delta_t^{DH} = \frac{\mu}{\theta \sigma^2 S_t}$$