Dynamically Characterising Time Series
With Relative Moving Moments

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5th of May 2024

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Abstract

We propose to dynamically characterise price series and identify trading opportunities by combining moving averages with price action to benefit from both approaches. Practically, we relate moving moments of the time series with specific asset prices, or statistical measures, used as anchors in the sliding window. The rolling moments of this normalised data (called relative moving moments) represent measures around the rolling distance between the data in the segment and its average. Even though we could achieve the same results by combining different technical indicators and extracting different pieces of information, following our approach, we have all the necessary information at a glance, so that we can directly perform market analysis and identify the right market conditions. To illustrate our approach, we generate realistic market prices, compute the relative moving moments and present our results. Finally, we use these results to discuss trading strategies and risk management.

Technical Indicators, Moving Average, Moments, Trading Strategies

1 Introduction

Quantitative trading consists in trading strategies based on quantitative analysis which rely on computer algorithms and programs based on mathematical models to (1) identify trading opportunities, and (2) capitalise on available trading opportunities (see Murphy [1999], Chan [2021]). The objective of quantitative trading is to calculate the optimal probability of executing a profitable trade. These models are first expressed as functions of information known about the future distribution of asset returns, and are then used for taking rational trading decisions (see Bloch [2023]). In this article, we focus on developing quantitative tools for dynamically characterising the price series and identifying the right market conditions for trading.

The principles of signal processing and digital control systems are well established, and statistical techniques are widely used for decomposing signals in various domains. It can be done in time domain (one-dimensional signals), spatial domain (multidimensional signals), frequency domain, wavelet domain, and signature domain. We need to find the domain that best represents the essential characteristics of the signal. A sequence of samples from a measuring device produces a temporal or spatial domain representation, whereas a discrete Fourier transform produces the frequency domain representation. In finance, the price time series evolve in the time-space domain and filtering
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Techniques are used, consisting of some linear transformation of a number of surrounding samples around the current input signal (see Bloch [2014]). Practitioners mostly consider moving averages to characterise the financial time series.

It is well known that most Technical Indicators (TI) (see Murphy [1999], Hill et al. [2000], Brabazon [2000]) rely on the computation of some simple moving averages (SMA) applied to prices, ranges, or returns, on a segment which is dynamically evolving via a sliding window mechanism. The moving average is seen as a normalisation and smoothing of the original price data, creating a constantly updated average price. They are calculated to identify the trend direction of a stock or to determine its support and resistance levels. For instance, a normalised moving average function is used to create bands to recognise overbought and oversold areas and identify possible entry points. However, since they are computed by using historical data, these averages exhibit a time lag. They depend on the window size: a faster moving average (short term or short lookback period) has less lag when compared to a slower moving average (long term or long lookback period). In moving average trading, each moving average indicator has its own pros and cons. The simple moving averages (SMA) moves much slower than other averages, keeping trades longer when there are short-lived price movements or price fluctuations. The exponential moving averages (EMA) reacts faster when the price is changing direction. For example, if the price retraces lower, the EMA will start turning down to indicate a change in the trading signal.

It is also well known that financial markets exhibit changing patterns (see Thomas [2000]) which traders try to identify on the fly to devise strategies (see Wang et al. [2007]). When performing Price Action Trading (PAT), traders need to identify trading patterns, entry and exit levels, stop-losses, and related observations. That is, they need to identify trading opportunities and devise appropriate strategies. For instance, decide whether a stock will form a double top to go higher, or whether it will drop further following a mean reversion. When a defined breakout scenario is met, trading opportunity exists in terms of breakout continuation. Traders rely on technical indicators (TIs) on charts to identify patterns that can help predict how a security will behave in the future and to time entry and exit points of trades. Since Price Action reflects all variables affecting that market for any given period of time, there is no need of using lagging price indicators such as Stochastics, MACD, RSI. Price movements and their ranges are the only useful information when performing PAT. As a result, volatility indicators developed to measure such price ranges.

In Sections (2) and (3), we propose to dynamically characterise price series and identify trading opportunities by combining moving averages (MAs) with price action (PA) to benefit from both approaches. Practically, we relate moving moments of the time series with specific asset prices, or statistical measures, used as anchors in the sliding window. It is well known that normalising the data of a segment with a constant and computing the sample moments is equivalent to directly normalising the moments with that constant. Hence, we let the constant be any element of the observed segment, or any statistical measure on that segment, and show that it acts as an anchor with respect to the other elements of the segment. As such, we can rewrite the data as a combination of backward returns with decreasing periods and forward returns with increasing periods. The rolling moments of this normalised data (called relative moving moments) represent measures around the rolling distance between the data in the segment and its average. We show that by carefully selecting the anchors, we can fully characterise the properties of the time series. Compared with other technical indicators, we get all the necessary information to dynamically characterise the time series and decide on the strategy to execute at a glance. In Section (4), we present a model to generate realistic market prices, display the relative moving moments (RMM), and analyse the resulting graphs. Then, we use these results to devise trading strategies and define risk management.

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2 The relative moving averages

2.1 Description

2.1.1 Formulation

We consider the time series \( X = \{ x_1, ..., x_n \}, x_t \in \mathbb{R} \), and generate a set of overlapping segments \( V = \{ v_1, ..., v_{n^*} \} \), where \( n^* = n - W + 1 \), by applying a sliding window of size \( W \) on \( X \). The kth segment is given by \( v_k = (x_k, ..., x_{k+W-1}) \). We normalise the kth segment with respect to any element of that segment, which we call the anchor. We denote \( v_k^{(i)}, i = 1, ..., W \), the normalised kth segment with respect to the ith element. This leads to rewriting each element of the segment in terms of some kind of returns as follows:

\[
v_k^{(i)} = \frac{1}{x_{k+i-1}} \cdot v_k = \frac{1}{x_{k+i-1}} \cdot (x_k, x_{k+1}, ..., x_{k+W-1}), i = 1, ..., W
\]

We compute the sample average (SA) of the normalised segment \( v_k^{(i)} \), getting

\[
\bar{v}_k^{(i)} = \frac{1}{W} \sum_{l=0}^{W-1} \frac{x_{k+l}}{x_{k+i-1}} - \alpha = \frac{x_k}{x_{k+i-1}} - \alpha \cdot \left( \frac{x_{k+1}}{x_{k+i-1}}, ..., \frac{x_{k+W-1}}{x_{k+i-1}} - \alpha \right) + \alpha
\]

for some constant \( \alpha \). Setting \( \alpha = 1 \), we get

\[
\bar{v}_k^{(i)} = \frac{1}{W} \sum_{l=0}^{W-1} \frac{x_{k+l}}{x_{k+i-1}} - 1 + 1 = \left( \frac{x_k}{x_{k+i-1}} - 1, \frac{x_{k+1}}{x_{k+i-1}} - 1, ..., \frac{x_{k+W-1}}{x_{k+i-1}} - 1 \right) + 1
\]

We can centre the indicator on zero as follows:

\[
\bar{v}_k^{(i-1)} = \bar{v}_k^{(i)} - 1 - \frac{1}{W} \sum_{l=0}^{W-1} \left( \frac{x_{k+l}}{x_{k+i-1}} - 1 \right) = \left( \frac{x_k}{x_{k+i-1}} - 1, \frac{x_{k+1}}{x_{k+i-1}} - 1, ..., \frac{x_{k+W-1}}{x_{k+i-1}} - 1 \right)
\]

where we are interested in the quantities \( \left( \frac{x_{k+l}}{x_{k+i-1}} - 1 \right) \), for \( l = 0, ..., W - 1 \), anchored around \( x_{k+i-1} \). Hence, this is equivalent to directly considering the modified kth segment

\[
v_k^{(i)} = \left( r_k^{(i)}, r_k^{(i)}, ..., r_k^{(i)} \right)
\]

where \( r_{k,l} = \left( \frac{x_{k+l}}{x_{k+i-1}} - 1 \right) \), and compute the SA

\[
\bar{v}_k^{(i)} = \frac{1}{W} \sum_{l=0}^{W-1} r_k^{(i)} - \bar{v}_k^{(i-1)}
\]

We call \( \bar{v}_k^{(i-1)} \) or \( \bar{v}_k^{(i)}, i = 1, ..., W \), the relative moving averages (RMA).
2.1.2 Some examples

For simplicity we present a few anchors corresponding to specific prices as well as statistical measures: first and last asset price in the segment, minimum, maximum, mode, quantiles.

1. The first element \( x_k \), getting

\[
v^{(1)}_k = \frac{1}{x_k} \cdot \frac{1}{x_k} \cdot (x_k, x_{k+1}, \ldots, x_{k+W-1}) = (1, \frac{x_{k+1}}{x_k}, \ldots, \frac{x_{k+W-1}}{x_k})
\]

The normalised series start at 1 and end up higher or lower than 1 depending on \( x_{k+W-1} \). We compute the sample average (SA) of the segment \( v^{(1)}_k \), getting

\[
\bar{v}^{(1)}_k = \frac{1}{x_k} \cdot \bar{v}_k = \frac{1}{W} \sum_{l=0}^{W-1} \frac{x_{k+l}}{x_k} - [1, \frac{x_{k+1}}{x_k}, \ldots, \frac{x_{k+W-1}}{x_k}]
\]

The indicator centred on zero becomes:

\[
\bar{v}^{(1)}_{(1)} - \bar{v}^{(1)}_k = 1 - \frac{1}{W} \sum_{l=0}^{W-1} \left( \frac{x_{k+l}}{x_k} - 1 \right) - [0, (\frac{x_{k+1}}{x_k} - 1), \ldots, (\frac{x_{k+W-1}}{x_k} - 1)]
\]

where we are interested in the quantities \( (\frac{x_{k+l}}{x_k} - 1) \), for \( l = 0, \ldots, W - 1 \), anchored around \( x_k \). Hence, this is equivalent to directly considering the modified kth segment

\[
v^{(1)}_{k,0} = (v^{(1)}_{k,0}, v^{(1)}_{k,1}, \ldots, v^{(1)}_{k,k+W-1})
\]

where \( v^{(1)}_{k,l} = (\frac{x_{k+l}}{x_k} - 1) \). Computing the SA, we get the RMA

\[
\bar{v}^{(1)}_{k,0} = \frac{1}{W} \sum_{l=0}^{W-1} v^{(1)}_{k,l} - \bar{v}^{(1)}_{(1)}
\]

From the definition of asset returns in Appendix (8.1), we let the simple (forward) return for \( d \)-period from date \( t \) to date \( t + d \) be

\[
R_{t,t+d} = \frac{\nabla^+_d x_t}{x_t} \text{ where } \nabla^+_d x_t = x_{t+d} - x_t
\]

so that \( r_{k,1} = R_{k,k+1} \). Thus, the kth segment \( v^{(1)}_{k,0} \) is a series of simple forward returns with increasing periods, and \( v^{(1)}_{k,0} \) is the sample average of these forward returns.

2. The last element \( x_{k+W-1} \), getting

\[
v^{(W)}_k = \frac{1}{x_{k+W-1}} \cdot (x_k, x_{k+1}, \ldots, x_{k+W-1}) = (\frac{x_k}{x_{k+W-1}}, \frac{x_{k+1}}{x_{k+W-1}}, \ldots, 1)
\]

The normalised series start above or below 1 depending on \( x_k \) and end up at 1. We compute the sample average (SA) of the segment \( v^{(W)}_k \), getting

\[
\bar{v}^{(W)}_k = \frac{1}{W} \sum_{l=0}^{W-1} \frac{x_{k+l}}{x_{k+W-1}} - [\frac{x_k}{x_{k+W-1}}, \frac{x_{k+1}}{x_{k+W-1}}, \ldots, 1]
\]
Again, rewriting each element as some kind of returns, we get the RMA

\[ y_k^{(W-1)} = y_k^W - 1 - \frac{1}{W} \sum_{l=0}^{W-1} \left( \frac{x_{k+l}}{x_{k+l+W-1}} - 1 \right) - \left[ \left( \frac{x_k}{x_{k+W-1}} - 1 \right), \left( \frac{x_{k+1}}{x_{k+W-1}} - 1 \right), \ldots, 0 \right] \]

This is equivalent to directly considering the modified kth segment

\[ v_{k,0}^{(W)} = (r_{k,0}^{(W)}, r_{k,1}^{(W)}, \ldots, r_{k,k+W-1}^{(W)}) \]

where \( r_{k,l}^{(W)} = \left( \frac{x_{k+l}}{x_{k+l+W-1}} - 1 \right) \), and computing the SA \( y_k^{(W)} \). In the spirit of asset returns (see Appendix (8.1)), we let the simple backward return for \( d \)-period from date \( t + d \) to date \( T \) be

\[ R_{t+d,T} = \frac{\sum_{t} x_t}{x_T} \text{ where } \sum_{t} x_t = x_{t+d} - x_T \]

so that \( r_{k,l} = R_{k+l,k+W-1} \). Thus, the kth segment \( v_{k,0}^{(W)} \) is a series of simple backward returns with decreasing periods, and \( v_{k,0}^{(W)} \) is the sample average of these backward returns.

3. The mid element \( x_{k+\lfloor W/2 \rfloor} \), getting the normalised segment

\[ v_{k}^{\lfloor W/2 \rfloor} = \frac{1}{x_{k+\lfloor W/2 \rfloor}} \cdot \left( x_{k}, x_{k+1}, \ldots, x_{k+W-1} \right) = \left( \frac{x_k}{x_{k+\lfloor W/2 \rfloor}}, \ldots, 1, \ldots, \frac{x_{k+W-1}}{x_{k+\lfloor W/2 \rfloor}} \right) \]

The normalised series start above or below 1 depending on \( x_k \), reach 1, and then end up above or below 1 depending on \( x_{k+W-1} \). We compute the sample average (SA) of the segment \( v_{k}^{\lfloor W/2 \rfloor} \), getting

\[ y_k^{\lfloor W/2 \rfloor} = \frac{1}{W} \sum_{l=0}^{W-1} \frac{x_{k+l}}{x_{k+\lfloor W/2 \rfloor}} = \left[ \left( \frac{x_k}{x_{k+\lfloor W/2 \rfloor}} - 1 \right), \ldots, 0, \ldots, \left( \frac{x_{k+W-1}}{x_{k+\lfloor W/2 \rfloor}} - 1 \right) \right] \]

Again, rewriting each element as some kind of returns, we get the RMA

\[ y_k^{\lfloor W/2 \rfloor} - 1 = \frac{1}{W} \sum_{l=0}^{W-1} \left( \frac{x_{k+l}}{x_{k+\lfloor W/2 \rfloor}} - 1 \right) - \left[ \left( \frac{x_k}{x_{k+\lfloor W/2 \rfloor}} - 1 \right), \ldots, 0, \ldots, \left( \frac{x_{k+W-1}}{x_{k+\lfloor W/2 \rfloor}} - 1 \right) \right] \]

This is equivalent to directly considering the modified kth segment

\[ v_{k,0}^{\lfloor W/2 \rfloor} = (r_{k,0}^{\lfloor W/2 \rfloor}, r_{k,1}^{\lfloor W/2 \rfloor}, \ldots, r_{k,k+W-1}^{\lfloor W/2 \rfloor}) \]

where \( r_{k,l}^{\lfloor W/2 \rfloor} = \left( \frac{x_{k+l}}{x_{k+W-1}} - 1 \right) \), and computing the SA \( y_k^{\lfloor W/2 \rfloor} \). For the first half of the segment we get a series of backward returns with decreasing periods, while for the second half we get a series of forward returns with increasing periods.

4. The min and max: We let \( x_{k,min} = \min v_k \) and \( x_{k,max} = \max v_k \) be the minimum and maximum values of the segment \( v_k \), respectively. Then, we get

\[ v_k^{(min)} = \frac{1}{x_{k,min}} \cdot (x_k, x_{k+1}, \ldots, x_{k+W-1}) \text{ and } v_k^{(max)} = \frac{1}{x_{k,max}} \cdot (x_k, x_{k+1}, \ldots, x_{k+W-1}) \]

\( \lfloor x \rfloor \) denotes the integer part of \( x \).

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Again, rewriting each element as some kind of returns, we get

\[ \pi_k^{\min/\max} = 1 - \frac{1}{W} \sum_{l=0}^{W-1} \left( \frac{x_{k+l}}{x_{\min/\max}} - 1 \right) - \left[ \left( \frac{x_k}{x_{\min/\max}} - 1 \right), ..., \left( \frac{x_{k+W-1}}{x_{\min/\max}} - 1 \right) \right] \]

This is equivalent to directly considering the modified kth segment

\[ \pi_{k,0}^{\min/\max} = (r_{k,0}^{\min/\max}, r_{k,1}^{\min/\max}, ..., r_{k,k+W-1}^{\min/\max}) \]

where \( r_{k,l}^{\min/\max} = \left( \frac{x_{k+l}}{x_{\min/\max}} - 1 \right) \), and computing the SA \( \tau_{k,0}^{\min/\max} \).

5. The median and mode: We let \( x_{\text{med}} \) be the median of the segment \( v_k \). Then, we get

\[ \pi_k^{\text{med}} = \frac{1}{x_{\text{med}}} \cdot (x_k, x_{k+1}, ..., x_{k+W-1}) \]

The RMA is

\[ \pi_k^{\text{med/med}} = 1 - \frac{1}{W} \sum_{l=0}^{W-1} \left( \frac{x_{k+l}}{x_{\text{med/med}}} - 1 \right) - \left[ \left( \frac{x_k}{x_{\text{med/med}}} - 1 \right), ..., \left( \frac{x_{k+W-1}}{x_{\text{med/med}}} - 1 \right) \right] \]

6. The quantiles: We let \( x_{\text{quant}} \) be the quantile of the segment \( v_k \). Then, we get

\[ \pi_k^{\text{quant}} = \frac{1}{x_{\text{quant}}} \cdot (x_k, x_{k+1}, ..., x_{k+W-1}) \]

The RMA is

\[ \pi_k^{\text{quant/quant}} = 1 - \frac{1}{W} \sum_{l=0}^{W-1} \left( \frac{x_{k+l}}{x_{\text{quant/quant}}} - 1 \right) - \left[ \left( \frac{x_k}{x_{\text{quant/quant}}} - 1 \right), ..., \left( \frac{x_{k+W-1}}{x_{\text{quant/quant}}} - 1 \right) \right] \]

### 2.2 Interpretation

There are two approaches for interpreting our TI: (1) as some kind of returns over increasing or decreasing time periods, (2) as the deviation of the data from the sample mean in a rolling window. In both cases, the Relative Moving Averages share the same properties.

#### 2.2.1 Some properties

For every normalised segment \( \tilde{u}_k^{(i)} \) with respect to the ith element, we have

\[ \pi_{k,0}^{(i)} = \pi_k^{(i)} - 1 = \frac{1}{x_{k+i-1}} \cdot \pi_k - 1, \ i = 1, ..., W \]

so that

1. if \( \pi_k > x_{k+i-1} \), then \( \pi_{k,0}^{(i)} > 0 \)
2. if \( \pi_k = x_{k+i-1} \), then \( \pi_{k,0}^{(i)} = 0 \)
3. if \( \pi_k < x_{k+i-1} \), then \( \pi_{k,0}^{(i)} < 0 \)
Further, if two different normalised mean of the k-segment are equal, we have

$$\bar{\tau}_{k,0}^{(i)} - \bar{\tau}_{k,0}^{(j)} = \frac{1}{x_{k+i-1}} \tau_k - \frac{1}{x_{k+j-1}} \tau_k \Rightarrow x_{k+i-1} = x_{k+j-1}$$

That is, in that segment, the two anchors $x_{k+i-1}$ and $x_{k+j-1}$ have the same value. Hence, when two relative moving averages cross, we know their anchors share the same value in that segment. We can deduce that (1) when two RMA converge the value of their respective anchors converge, (2) when two RMA diverge the value of their respective anchors diverge.

### 2.2.2 Types of returns

In the former, we can interpret our TI as follows:

1. The first element $x_k$: Given $\bar{\tau}_{k,0}^{(1)}$, if all the elements $\{x_{k+1}, \ldots, x_{k+W-1}\}$ are close to $x_k$, then $\bar{\tau}_{k,0}^{(1)} \approx 0$. If $\bar{\tau}_{k,0}^{(1)} > 0$ then the current values are higher than the past values, indicating a positive acceleration of market prices. On the other hand, if $\bar{\tau}_{k,0}^{(1)} < 0$ then the current values are lower than the past values, indicating a negative acceleration of market prices.

2. The last element $x_{k+W-1}$: Given $\bar{\tau}_{k,0}^{(W)}$, if all the elements $\{x_k, \ldots, x_{k+W-2}\}$ are close to $x_{k+W-1}$, then $\bar{\tau}_{k,0}^{(W)} \approx 0$. If $\bar{\tau}_{k,0}^{(W)} > 0$ then the past values are higher that the current values, indicating a negative acceleration of market prices. On the other hand, if $\bar{\tau}_{k,0}^{(W)} < 0$ then the past values are lower that the current values, indicating a positive acceleration of market prices.

3. The mid element $x_{k+\left(\frac{W}{2}\right)}$: If the returns on both sides of $x_{k+\left(\frac{W}{2}\right)}$ are positive, $\bar{\tau}_{k,0}^{(\frac{W}{2})} > 0$ and we have a positive reversal at $x_{k+\left(\frac{W}{2}\right)}$. On the other hand, if the returns on both sides of $x_{k+\left(\frac{W}{2}\right)}$ are negative, $\bar{\tau}_{k,0}^{(\frac{W}{2})} < 0$ and we have a negative reversal at $x_{k+\left(\frac{W}{2}\right)}$.

4. The min and max: Given $\bar{\tau}_{k,0}^{(\min\max)}$, if all the elements in $v_k$ are close to the min/max, then $\bar{\tau}_{k,0}^{(\min\max)} \approx 0$. If $\bar{\tau}_{k,0}^{(\min)} > 0$, then $\tau_k > x_{\min}^k$, while if $\bar{\tau}_{k,0}^{(\max)} < 0$, then $\tau_k < x_{\max}^k$.

5. The median and mode: Given $\bar{\tau}_{k,0}^{(\text{med}/\text{mod})}$, if all the elements in $v_k$ are close to the medianmode, then $\bar{\tau}_{k,0}^{(\text{med}/\text{mod})} \approx 0$. If $\bar{\tau}_{k,0}^{(\text{med}/\text{mod})} > 0$, then $\tau_k > x_{\text{med}/\text{mod}}^k$, while if $\bar{\tau}_{k,0}^{(\text{med}/\text{mod})} < 0$, then $\tau_k < x_{\text{med}/\text{mod}}^k$.

6. The quantiles: Given $\bar{\tau}_{k,0}^{(\text{quant})}$, if all the elements in $v_k$ are close to the quantile, then $\bar{\tau}_{k,0}^{(\text{quant})} \approx 0$. If $\bar{\tau}_{k,0}^{(\text{quant})} > 0$, then $\tau_k > x_{\text{quant}}^k$, while if $\bar{\tau}_{k,0}^{(\text{quant})} < 0$, then $\tau_k < x_{\text{quant}}^k$.

### 2.2.3 Relative deviation

In the latter, we can interpret our TI as follows:

1. The first element $x_k$: it gives long term directions

   1.1 When $\bar{\tau}_{k,0}^{(1)} > 0$, then $x_k$ is below the sample mean. Hence, the series is increasing with respect to the first element.
   1.2 When $\bar{\tau}_{k,0}^{(1)} \approx 0$, then $x_k$ is very close to the sample mean.
   1.3 When $\bar{\tau}_{k,0}^{(1)} < 0$, then $x_k$ is above the sample mean. Hence, the series is decreasing with respect to the first element. The larger $\bar{\tau}_{k,0}^{(1)}$, the larger is the deviation from the sample mean.
2. The last element $x_{k+4W-1}$: it gives current directions

2.1 When $\tau_{k,0}^{(W)} > 0$, then $x_{k+4W-1}$ is below the sample mean. Hence, the series is decreasing with respect to the last element.

2.2 When $\tau_{k,0}^{(W)} \approx 0$, then $x_{k+4W-1}$ is very close to the sample mean.

2.3 When $\tau_{k,0}^{(W)} < 0$, then $x_{k+4W-1}$ is above the sample mean. Hence, the series is increasing with respect to the last element. The larger $\tau_{k,0}^{(1)}$, the larger is the deviation from the sample mean.

3. The mid element $x_{k+4W-1}$:

3.1 When $\tau_{k,0}^{(1)} > 0$, then $x_{k+4W-1}$ is below the sample mean. We need to know about $\tau_{k,0}^{(1)}$ and $\tau_{k,0}^{(W)}$ to infer the properties of the series.

3.2 When $\tau_{k,0}^{(1)} \approx 0$, then $x_{k+4W-1}$ is very close to the sample mean.

3.3 When $\tau_{k,0}^{(1)} < 0$, then $x_{k+4W-1}$ is above the sample mean. We need to know about $\tau_{k,0}^{(1)}$ and $\tau_{k,0}^{(W)}$ to infer the properties of the series.

4. The min and max: Note, $\tau_{k,0}^{(min)}$ is always positive and it decreases towards zero if $x_{k}^{min}$ decreases or if $x_{k}^{min}$ increases. Conversely, $\tau_{k,0}^{(max)}$ is always negative and it increases towards zero if $x_{k}$ increases or if $x_{k}^{max}$ decreases.

5. The median and mode: They can be seen as a measure of skewness. If $\tau_{k,0}^{(med/\text{mod})} > 0$, the mean is greater than the median/mode, we have positive skew. Conversely, if $\tau_{k,0}^{(med/\text{mod})} < 0$, the mean is smaller than the median/mode, we have negative skew. When $\tau_{k,0}^{(med/\text{mod})} \approx 0$, the mean and the median/mode coincide and the distribution is Gaussian.

6. The quantiles: $Q_1$ and $Q_3$ can be seen as a measure of fat tails. When $\tau_{k,0}^{(quant)} \approx 0$, the mean and the quantile coincide.

2.3 Characterising the series

Using the above results on the relative deviation from the mean, we are now going to characterise the time series. For simplicity, we consider three normalised sample mean: $\tau_{k,0}^{(1)}$, $\tau_{k,0}^{(W)}$, and $\tau_{k,0}^{(1/W)}$ anchored on the first element, last element, and middle element, respectively.

- The mid element $x_{k+4W-1}$:
  - When $\tau_{k,0}^{(W)} > 0$: if $\tau_{k,0}^{(W)} < 0$, then $x_{k+4W-1} < \bar{x}$ and $x_{k+4W-1} > \bar{x}$. The last part of the time series is increasing. We are in a local up-trend.
  - When $\tau_{k,0}^{(W)} \approx 0$:
    * if $\tau_{k,0}^{(1)}$ and $\tau_{k,0}^{(W)}$ are far from 0 and converge to 0, the time series is flat with low volatility. We move in a Squeeze.
    * if $\tau_{k,0}^{(1)}$ and $\tau_{k,0}^{(W)}$ are close to 0 and suddenly diverge from 0, the time series burst away from its level with high volatility. We move in a Burst.
  - When $\tau_{k,0}^{(W)} < 0$: if $\tau_{k,0}^{(W)} > 0$, then $x_{k+4W-1} > \bar{x}$ and $x_{k+4W-1} < \bar{x}$. The last part of the time series is decreasing. We are in a local down-trend.
• The first and last element, $x_k$ and $x_{k+W-1}$:
  - If $\bar{\tau}_{k,0}^{(1)} > 0$ and $\bar{\tau}_{k,0}^{(W)} < 0$, then $x_k < \bar{\tau}_k$ and $x_{k+W-1} > \bar{\tau}_k$. The time series is increasing, we are in a persistent up-tend.
  - If $\bar{\tau}_{k,0}^{(1)} < 0$ and $\bar{\tau}_{k,0}^{(W)} > 0$, then $x_k > \bar{\tau}_k$ and $x_{k+W-1} < \bar{\tau}_k$. The time series is decreasing, we are in a persistent down-tend.

• The first and mid element, $x_k$ and $x_{k+W/2}$:
  - If $\bar{\tau}_{k,0}^{(1)} > 0$ and $\bar{\tau}_{k,0}^{(W)} < 0$, then $x_k < \bar{\tau}_k$ and $x_{k+W/2} > \bar{\tau}_k$. The time series started increasing and it reverts (decreases) at the mid. The second part depends on $\bar{\tau}_{k,0}^{(W)}$.
  - If $\bar{\tau}_{k,0}^{(1)} < 0$ and $\bar{\tau}_{k,0}^{(W)} > 0$, then $x_k > \bar{\tau}_k$ and $x_{k+W/2} < \bar{\tau}_k$. The time series started decreasing and it reverts (increases) at the mid. The second part depends on $\bar{\tau}_{k,0}^{(W)}$.

• The three elements, $x_k, x_{k+W-1}$, and $x_{k+W/2}$:
  - If $\bar{\tau}_{k,0}^{(1)} > 0$ (slightly positive), $\bar{\tau}_{k,0}^{(W)} < 0$ (slightly negative), and $\bar{\tau}_{k,0}^{(W/min)} \approx 0$, then the time series is increasing in a constant manner. We are in a steady up-tend.
  - If $\bar{\tau}_{k,0}^{(1)} < 0$ (slightly negative), $\bar{\tau}_{k,0}^{(W)} > 0$ (slightly positive), and $\bar{\tau}_{k,0}^{(W/min)} \approx 0$, then the time series is decreasing in a constant manner. We are in a steady down-tend.

• The min and max:
  - When $\bar{\tau}_{k,0}^{(1)} < 0$ and $\bar{\tau}_{k,0}^{(W)} \approx \bar{\tau}_{k,0}^{(max)}$, the first element $x_k$ is around $x_{k}^{min}$. Conversely, when $\bar{\tau}_{k,0}^{(1)} > 0$ and $\bar{\tau}_{k,0}^{(W)} \approx \bar{\tau}_{k,0}^{(max)}$, the first element $x_k$ is around $x_{k}^{max}$. Conversely, when $\bar{\tau}_{k,0}^{(W)} < 0$ and $\bar{\tau}_{k,0}^{(W)} \approx \bar{\tau}_{k,0}^{(max)}$, the current value $x_{k+W-1}$ is around $x_{k+1}^{max}$. Conversely, when $\bar{\tau}_{k,0}^{(W)} > 0$ and $\bar{\tau}_{k,0}^{(W)} \approx \bar{\tau}_{k,0}^{(min)}$, the current value $x_{k+W-1}$ is around $x_{k+1}^{min}$.

3 The relative moving moments

3.1 The first four moments

Rather than computing the sample mean on the normalised segments $v_{k,0}^{(i)}$, $i = 1, ..., W$, we rely on the properties of the first two moments (see Appendix (8.2.2)), and compute the sample mean on the segments $v_k$ which we scale with the anchor. We get

$$\bar{\mu}_{k(i)} = \frac{1}{x_{k+i-1}} \cdot \Sigma v_k - \bar{\tau}_{k,0}^{(i)}, \ \ i = 1, ..., W$$

where $\bar{\mu}_k$ is the sample mean of the segment $v_k$. We can also compute the sample variance of the normalised segments, getting

$$Var(v_k^{(i)}) - \sigma_{k(i)}^2 = \frac{1}{x_{k+i-1}} \cdot Var(v_k) - \frac{1}{x_{k+i-1}} \sigma_k^2 - Var(v_{k,0}^{(i)})$$

We let $\mu_n^k - \mu_n(v_k)$ be the nth central moment of the segment $v_k$ and $\sigma_n^2$ be its variance. We also let $\mu_n^{k(i)} - \mu_n(v_{k(i)})$ be the nth central moment of the normalised segment $v_k^{(i)}$ and $\sigma_n^{k(i)}$ be its variance. From the properties of the standardised moments (see Appendix (8.2.1)), the sample skew of the normalised segments $v_k^{(i)}$ is
We recover exactly the skew and kurtosis for ing models (CRSMs). We will then use the results from Section (2.3) to characterise the time series. We are going to validate the relative moving moments (RMM) on trajectories generated with complex regime switching models (CRSMs).

\[ S_k^{(i)} = \frac{\mu_k^{(i)}}{\sigma_{k,(i)}^2} x_{k+i-1}^3 - \frac{x_{k+i-1}^3}{\sigma_{k,(i)}^2} \mu_k^{(i)} \frac{1}{\sigma_{k,(i)}^2} S_k, \ i = 1, \ldots, W \]

and the sample kurtosis of the normalised segments is

\[ K_k^{(i)} = \frac{\mu_k^{(i)}}{\sigma_{k,(i)}^2} - \frac{x_{k+i-1}^4}{\sigma_{k,(i)}^2} \mu_k^{(i)} \frac{1}{\sigma_{k,(i)}^2} K_k, \ i = 1, \ldots, W \]

where \( \mu_j^{(i)} \) is the jth central moment of the normalised segments \( v_k^{(i)} \), and \( \mu_j^k \) is the jth central moment of the segments \( v_k \).

### 3.2 The relative central moments

Given the normalised segments \( v_k^{(i)}, i = 1, \ldots, W \), the standardised moment of degree \( n \), denoted \( \gamma_n(X) = \frac{\mu_n}{\sigma^n} \), where \( \mu_n \) is the nth central moment and \( \sigma^2 \) is the variance, are equal to those computed on the regular segment \( v_k \). This is because these moments scale as \( x^n \), that is, \( \mu_n(AX) = \lambda^n \mu_n(X) \). Hence, they are homogeneous functions of degree \( n \), so that the standardised moments are scale invariant. As a result, we cannot use them directly and choose to work with relative central moments. The nth relative central moments of the normalised segments are given by

\[ \mu_n(v_k^{(i)}) = \frac{1}{x_{k+i-1}^n} \mu_n(v_k) \]

For example, the third and fourth central moments are

\[ \mu_3(v_k^{(i)}) = \frac{1}{x_{k+i-1}^3} \mu_3(v_k) \quad \text{and} \quad \mu_4(v_k^{(i)}) = \frac{1}{x_{k+i-1}^4} \mu_4(v_k) \]

We observe that the order of magnitude of \( \mu_3^{(i)} \) and \( \mu_4^{(i)} \) is smaller than that of \( \gamma_3(v_k) \) and \( \gamma_4(v_k) \). Since we want to display these measures in a graph with other metrics, we want to rescale these relative moments so that their order of magnitude is close to that of \( \gamma_3(v_k) \) and \( \gamma_4(v_k) \), respectively. That is,

\[ \alpha \cdot \mu_3(v_k^{(i)}) \approx \gamma_3(v_k) \quad \text{and} \quad \beta \cdot \mu_4(v_k^{(i)}) \approx \gamma_4(v_k) \]

We recover exactly the skew and kurtosis for \( \alpha = x_{k+i-1}^3 / \sigma_k^2 \) and \( \beta = x_{k+i-1}^4 / \sigma_k^2 \). Since we are working on prices, \( \sigma_k > 1 \) so that \( \alpha < x_{k+i-1}^3 \) and \( \beta < x_{k+i-1}^4 \). For simplicity, we take the square-root of \( x_{k+i-1}^n, \ n = 3, 4 \), getting \( \alpha = x_{k+i-1}^{3/2} \) and \( \beta = x_{k+i-1}^{2} \). Hence, the rescaled third and fourth relative central moments on \( v_k^{(i)} \) become:

\[ \mu_3(v_k^{(i)}) = \frac{1}{x_{k+i-1}^{3/2}} \mu_3(v_k) \quad \text{and} \quad \mu_4(v_k^{(i)}) = \frac{1}{x_{k+i-1}^{2}} \mu_4(v_k) \quad (3.1) \]

### 4 Results

We are going to validate the relative moving moments (RMM) on trajectories generated with complex regime switching models (CRSMs). We will then use the results from Section (2.3) to characterise the time series.
4.1 CRSM

For testing purposes, we develop a simulation method for asset returns based on complex regime-switching models (CRSMs). The stock price process can switch between $K$ regimes randomly, where each regime is characterised by a complex process with a set of model parameters. In addition, this simulation method accepts a specific set of time points as extra input and then generates jumps of a pre-set magnitude at these time points.

Let the continuous time set be $T_C = [0, T]$ and the trading time set be $T = \{t_0 < t_1 < \ldots < t_n < T\}$ with discretisation time $\delta = \frac{T}{n}$. Further, consider the non-uniform time set $T' = \{t_1', \ldots, t_m'\}$. We merge the two time sets $T$ and $T'$ to get $\bar{T}_m = T \cup T'$. For a fixed state, assume $S$ follows a complex model on the time grid $T$ and exhibits jumps on the time grid $T'$. We change states based on a transition probability matrix (with element $p_{ij}$ to go from state $i$ to state $j$).

For $K$ states, the dynamics of $S$, under the $\mathbb{P}$-measure, are given by

$$
\frac{dS_t}{S_t} = \gamma d\varphi_t + \left\{ \begin{array}{ll}
\left( I_{(a^{(1)} \leq \varphi_t < b^{(1)})} - q^{(1)} \right) I_{(a^{(1)} \leq \varphi_t < b^{(1)})} dt + \sigma^{(1)} I_{(a^{(1)} \leq \varphi_t < b^{(1)})} d\widetilde{W}^{(1)}_{H,S}(t) + (J^{(1)}(1) - 1) dN^{(1)}_t \\
\ldots \\
\left( I_{(a^{(K)} \leq \varphi_t < b^{(K)})} - q^{(K)} \right) I_{(a^{(K)} \leq \varphi_t < b^{(K)})} dt + \sigma^{(K)} I_{(a^{(K)} \leq \varphi_t < b^{(K)})} d\widetilde{W}^{(K)}_{H,S}(t) + (J^{(K)}(1) - 1) dN^{(K)}_t
\end{array} \right.
$$

for some jump process $\varphi_t$ and jump size $\gamma$ (which can be random). Both the short rate $r_t$ and the instantaneous volatility $\sigma_t$ follow some stochastic processes. Further, $I_{(a, b)}$ is an indicator function being 1 if the event is in $(a, b)$, else it is 0.

$\widetilde{W}_H$ is a fractional Brownian motion (FBM) with Hurst coefficient $H$ taking value in $(0, 1)$. Increments of FBM are not independent: if $H > \frac{1}{2}$ there is positive autocorrelation, while if $H < \frac{1}{2}$ there is negative autocorrelation. Hence, the process is self-similar. If $H > \frac{1}{2}$ the process exhibits long-range dependence. Here, the rate $r_t$ and the market-price-of-risk $\lambda_t$ follow an FBM-OU process, and the volatility $\sigma_t$ follow an FBM-square-root process. We have $<d\widetilde{W}_{H,S}, d\widetilde{W}_{H,R}> = \rho_{S,R}$, $<d\widetilde{W}_{H,S}, d\widetilde{W}_{H,\lambda}> = \rho_{S,\lambda}$, and $<d\widetilde{W}_{H,S}, d\widetilde{W}_{H,\sigma}> = \rho_{S,\sigma}$.

Let $x_t$ be a FBM-OU-process with dynamics

$$
dx_t = a_x (b_x - x_t) dt + \sigma_x d\widetilde{W}_{H,x}(t), \quad x_0 = x
$$

where $a_x > 0$ is the speed-of-mean-reversion, $b_x$ is the infinite value, and $\sigma_x > 0$ is the volatility. Also, let $y_t$ be an FBM-square-root-process with dynamics

$$
dy_t = a_y (b_y - y_t) dt + \sigma_y \sqrt{y_t} d\widetilde{W}_{H,y}(t), \quad y_0 = y
$$

where $a_y > 0$ is the speed-of-mean-reversion, $b_y$ is the infinite value, and $\sigma_y > 0$ is the volatility. We have $<d\widetilde{W}_{H,x}, d\widetilde{W}_{H,y}> = \rho_{x,y} dt$.

Finally, $N_t$ denotes a Poisson process with positive jump size $J$ (we ignore the compensator). The jump process has intensity $\lambda_J$, which is the mean number of arrivals per unit time. We assume that $d\widetilde{W}_t$ and $dN_t$ are independent. In the Merton model, the size of the jump $J$ is a random variable such that $\log(J)$ is normally distributed with constant mean and constant variance $\gamma_J^2$. We let $\kappa = E[J - 1]$ and denote $\mu_J = \log(1 + \kappa) - \log(E[J])$, so that $E[J] = e^{\mu_J}$. As a result, we must have

$$
J = e^{\mu_J - \frac{1}{2} \gamma_J^2 + \gamma_J Z_t}
$$

where $Z_t$ follows a standard normal law. Thus, the expected value of $J$ is $E[J] = e^{\mu_J}$.
4.2 Settings

4.2.1 Model parameters

For simplicity we consider a single regime $K = 1$ and focus on a diffusion model with no restriction, that is, $a^{(1)} = 0$ and $b^{(1)} = \infty$. We assume an OU process for the short rate $r_t$ and a square-root process for the volatility $\sigma_t$. The model parameters are

- Vasicek: $a_r = 3, b_r = 0.07, \sigma_r = 0.35, \rho_{rS} = 0.7, r_0 = 0.03$
- Heston: $a_v = 5, b_v = (0.35)^2, \sigma_v = 0.75, \rho_{vS} = -0.7, v_0 = (0.2)^2$

The initial spot price is $S_0 = 100$, the dividend rate is $q = 0$, the discretisation time is $\delta = 1$ day, the maturity is $T = 5$ years, and the window size is $W = 200$ days.

4.2.2 Displaying results

We display in Figure (1) the price time series and its associated moving average. We also display the rolling RMA for the specific anchors discussed above. In Figure (2), we display the price time series and its associated moving average as well as the rolling relative central moments. We display the volatility for the three anchors: first, last, and mid elements. We also display the volatility of the prices (blue curve) which we normalise with $S_0$ for comparison. Then, we display the relative central third and fourth moments (see Equation (3.1)), which we compare against the skew and kurtosis (blue curves), respectively. Note, in order to get the relative central moments within $[0, 1]$, we further scale these moments by dividing them with 6 and 24, respectively. We also divide the skew by 6 and the kurtosis by 24.

The labelling of the curves is as follows:

- Blue curve: price time series
- Brown curve: moving average
- Yellow curve: RMA with first element $x_k$ as anchor
- Orange curve: RMA with mid element $x_{k+[W/2]}$ as anchor
- Limegreen curve: RMA with last element $x_{k+W-1}$ as anchor
- Dodgerblue curve: RMA with minimum element $x_{\min}$ as anchor. We call this curve the Lowest Low (LL) curve.
- Aqua curve: RMA with maximum element $x_{\max}$ as anchor. We call this curve the Highest High (HH) curve.
- Purple curve: RMA with $Q2$ element $x_{Q2}$ as anchor
- Hotpink curve: RMA with $Q1$ element $x_{Q1}$ as anchor
- Darkviolet curve: RMA with $Q3$ element $x_{Q3}$ as anchor
Figure 1: Prices and rolling RMA for some specific anchors.
4.3 Analysis

Since the price time series oscillate around the moving average (MA) in a sigmoid fashion, the relative moving averages (RMA) also oscillate in a sigmoid fashion, but in a less smooth manner due to the randomness of the anchors. This is illustrated in Figure (1). When at time $t$, the green indicator relates the moving average (MA) to the current asset price $S_t$ (we denote $S_{now}$). As such, it reacts instantaneously to changes in market prices. The orange indicator relates the MA to the asset price in the middle of the segment $S_{t-W}$ (we denote $S_{mid}$), and the yellow indicator relates the MA to the first price in the segment $S_{t-W+1}$ (we denote $S_{past}$). Both the orange and yellow indicators relate the MA to
past asset prices with increasing distances. Hence, when at time $t$, the distances between the MA and these prices tell us information about the trajectory of the underlying asset. Similarly, the brown indicator and aqua indicator relate the MA to the minimum (lowest low) and the maximum (highest high), respectively, in the current segment. Combining these five indicators we can characterise the price time series.

We briefly explain the behaviour of the RMA displayed in Figure (1) over a few time periods:

- In the time period $[200, 400]$ the asset prices oscillate around 110 and the MA slightly increase from 110 till 118. At 200, $S_{now} = 130$ and the green curve is at the max slightly before $-0.2$. Since both the orange and yellow curves are around 0, we can deduce that $S_{past}$ was around 110 two hundred days ago and that $S_{mid}$ was around 110 one hundred days ago. Then the spot price decreases sharply to cross the MA, the green curve simultaneously increase to cross zero. Meanwhile, $S_{mid}$ must have sharply decreased and quickly bounce back because the orange curve increased to hit the LL (the mid value is at the lowest) and went back to cross the green and yellow curve at zero (around 110).

- In the time period $[400, 600]$ the asset prices are around 118 and suddenly drop to 80, oscillate at that level and reach 70 before reverting to 80 It then jumps to 90 oscillate and quickly decrease to 70 before reverting to 80 and finally reaching 70. At the start of this period the spot is at the mean so that the green curve is at zero. Further, both the orange and yellow curves are around zero so that $S_{now} \approx S_{mid} \approx S_{past} \approx 110$. The green curve cross the LL realising a new low slightly above 0.4, while the orange and yellow curves reach the HH between $-0.1$ and $-0.2$. When the spot prices reach 70 for the first time, the green curve reaches 0.5 producing a new LL. As the spot prices decrease, the MA decreases to that the HH decrease (away from zero) and the LL also decrease (towards zero). Further, $S_{past}$ stays at the HH, while the orange curve meets the green one and $S_{mid} \approx S_{now}$. It indicates that they both were at 70 and 80 forming supports at these levels.

- In the period $[1000, 1200]$ the asset prices are around 80 sharply jump to 70 before reverting to 80 to quickly increase to pass 110 before sharply decreasing to 90 and reverting back to 110. The three indicators are roughly around zero (at the mean) and as soon as the prices jump from 80 to 70 the green curve jumps to 0.2 creating a new LL. It then revert back to zero, where the orange and yellow curves hardly moved (oscillation around zero). Then, as prices sharply increase to 110, the green curve reaches $-0.2$ creating a new HH, while the orange and yellow curves increase to 0.1 reflecting the sudden increase in prices. As prices from 110 to 90 they cross the average around 95 where the three indicators cross in zero. The price reverting from 90 back to 110 is indicated by the orange and green curve going to $-0.1$, while the yellow curve goes from zero to 0.2. The green curve at $-0.1$ indicates that $S_{now} = 110$, while the orange curve at $-0.1$ indicates that $S_{mid} = 110$, but the yellow at 0.2 indicates that $S_{past} \approx 80$.

5 Strategies and risk management

Using the results from Section (4), we can devise trading strategies and define risk management. Clearly, we could achieve the same results by combining different technical indicators, extracting different pieces of information from them, normalising the information, and identifying the right market conditions for trading. Fortunately, following our approach, we have all the necessary normalised information at a glance, so that we can directly perform analysis and identify the right market conditions.

Depending on the risk-return appetite from traders, there exists a large number of trading strategies and associated risk management. Focusing on the RMA in Figure (1), we are only going to hint at some simple strategies and their associated risk management. First, we note that the RMA are all contained within the Lowest Low curve (Dodgerblue curve) and the Highest High curve (Aqua curve). Second, these curves are dynamic, evolving with the distance between the MA curve and the price time series. They expend when that distance increases and contract when it decreases. Hence, when in a time period, we can use the LL and HH curves from the previous time period as hard
stop loss and profit taking (depending on the sign of the position). These risk limits will automatically adjust as time evolve.

We are now going to consider a few time periods in Figure (1) and walk the readers through a few possible strategies and risk management:

- Short position: This is illustrated in the time period $[400, 600]$. The three curves are at zero just before 400 at $S_{now} \approx 115$. We go short as soon as the limegreen curve jumps upwards (with spot price possibly around $110/105$), crossing the LL (at 0.2), and both the orange and yellow curves sharply moves downwards, possibly hitting the HH (above $-0.2$). $S_{past} \approx S_{mid}$ are at the maximum while $S_{now}$ is at the minimum. As long as both the LL and HH decrease, we are in a down trend. The curves meet again around zero just after 600 where the spot price is around 80.

- Long position: This is illustrated in the time period $[1050, 1150]$. The three curves are at zero around 1050 at $S_{now} \approx 90$. We go long as soon as the limegreen curve jumps downwards, crossing the HH (below $-0.1$), and both the orange and yellow curves sharply moves upwards, possibly hitting the LL. $S_{past} \approx S_{mid}$ are moving towards the maximum while $S_{now}$ is at the minimum. As long as both the LL and HH increase, we are in an up trend. The curves meet again around zero around 1150 where the spot price is around 90.

- The time period $[800, 1000]$. Both the LL and HH curves are concentrated around zero, with values 0.1 and $-0.1$, respectively, indicating a squeeze where the spot prices oscillate around the mean with small amplitudes.

- When execution is fast enough (possibly HF), we can consider mean-reverting strategies within a trading cycle.

- Short position: This is illustrated in the time period $[400, 600]$. As in the previous case, we go short as soon as the limegreen curve jumps upwards, crossing the LL (at 0.2) (previous period support), and both the orange and yellow curves sharply moves downwards, possibly hitting the HH (above $-0.2$). We get a first new support when the limegreen curve reaches 0.4 and another one when it reaches 0.5. We can enter mean-reversion strategy by observing that the orange curve is reverting between zero and the HH. If we are fast enough, we exit at $0.5$ and get long at 0.5 and exit at 0.4. Otherwise, we exit when the limegreen curve get back to 0.4 (coming from 0.5) and the orange curve gets back to the HH (coming from 0.1). A natural stop loss is the LL of the previous period at 0.2. A re-entry point is when the limegreen curve is at 0.1 and jumps upwards to hit the LL at 0.3. Note, the yellow curve is still oscillating at the HH, still indicating a downward trend. However, the orange curve is now positive at 0.1 and the median (Q2) is positive, indicating the end of that trend. Hence, we can exit there. Before 600, the limegreen curves sharply decrease and returns to zero while the yellow curve increases and also returns to zero. The three curves meet again in zero just after zero.

- Long position: This is illustrated in the time period $[1050, 1300]$. As in the previous case, we go long as soon as the limegreen curve jumps downwards, crossing the HH (below $-0.1$), and both the orange and yellow curves sharply moves upwards, possibly hitting the LL. We get a first new support when the limegreen curve reaches $-0.1$ and another one when it reaches $-0.2$. Around 1100, $S_{mid} \approx 70$ is at the minimum at 0.25 while $S_{now} \approx 110$ is at the maximum at $-0.2$, but $S_{past} \approx 90$ is only at 0.1. The former indicates past reversal around the mean and the latter a sudden sharp upward jump from the spot price. Hence, assuming mean-reversion, we exit when the green curve hit the resistance at $-0.2$, and go short until it reaches 0 where we unwind position. The three curves meet again at zero before 120 around 95 and diverge. Both the orange curve and green curve sharply decrease to the HH around $-0.15$ while the yellow curve increases to hit the LL around 0.2 (previous support). Since we have observed past reversals around
the mean, we exit when the green curve hits the HH (resistance at $-0.2$) and go long until it reaches zero where we unwind position. The three curves meet again in zero after 1200 around 100 and diverge. Both the LL and HH curves are increasing, indicating an upwards trend.

6 Conclusion

We have shown that by combining moving averages with price action to benefit from both approaches, we could dynamically characterise price series and identify trading opportunities. We obtained a range of technical indicators by relating the moving moments of the time series with specific asset prices. These indicators provide all the necessary information for characterising time series at a glance. Hence, we can directly perform market analysis and identify the right market conditions. Further, these indicators can be used as input in Deep Networks to infer optimal trading strategy.
7 Appendix
8 Miscellaneous
8.1 Asset returns

Return series are easier to handle than price series due to their more attractive statistical properties and to the fact that they represent a complete and scale-free summary of the investment opportunity (see Campbell et al. [1997]). Expected returns need to be viewed over some time horizon, in some base currency, and using one of many possible averaging and compounded methods. Holding the asset for one period from date \( t \) to date \((t + 1)\) would result in a simple gross return

\[
1 + R_{t,t+1} = \frac{S_{t+1}}{S_t}
\]  

(8.2)

where the corresponding one-period simple net return, or simple return, \( R_{t,t+1} \) is given by

\[
R_{t,t+1} = \frac{S_{t+1} - S_t}{S_t} - 1
\]

More generally, we let \( R_{t,d,t} = \frac{\nabla_d S_t + D_{t-d,t}}{S_{t-d}} \) be the discrete return of the underlying process where \( \nabla_d S_t = S_t - S_{t-d} \) with period \( d \) and where \( D_{t-d,t} \) is the dividend over the period \([t-d,t]\). For simplicity we will only consider dividend-adjusted prices with discrete dividend-adjusted returns \( R_{t-d,t} = \frac{\nabla_d S_t}{S_{t-d}} \). Hence, holding the asset for \( d \) periods between dates \( t-d \) and \( t \) gives a \( d \)-period simple gross return

\[
1 + R_{t-d,t} = \frac{S_t}{S_{t-d}} - \frac{S_{t-1}}{S_{t-1}} \times \ldots \times \frac{S_{t-d+1}}{S_{t-d}} = (1 + R_{t-1,t})(1 + R_{t-2,t-1})\ldots(1 + R_{t-d,t-d+1}) - \prod_{j=1}^{d}(1 + R_{t-j,t-j+1})
\]

(8.3)

so that the \( d \)-period simple gross return is just the product of the \( d \) one-period simple gross returns, which is called a compound return. Holding the asset for \( d \) years, then the annualised (average) return is defined as

\[
R_{t-d,t}^A = \left( \prod_{j=1}^{d}(1 + R_{t-j,t-j+1}) \right)^{\frac{1}{d}} - 1
\]

which is the geometric mean of the \( d \) one-period simple gross returns involved and can be computed by

\[
R_{t-d,t}^A = e^{\frac{1}{d} \sum_{j=1}^{d} \ln (1 + R_{t-j,t-j+1})} - 1
\]

It is simply the arithmetic mean of the logarithm returns \((1 + R_{t-j,t-j+1})\) for \( j = 1, \ldots, d \) which is then exponentiated to return the computation to the original scale. As it is easier to compute arithmetic average than geometric mean, and since the one-period returns tend to be small, one can use a first-order Taylor expansion \(^2\) to approximate the annualised (average) return

\(^2\)since \( \log (1 + x) \approx x \) for \(|x| \leq 1 \)
\[ R_{t-d,t}^A \approx \frac{1}{d} \sum_{j=1}^{d} R_{t-j,t+j+1} \]

Note, the arithmetic mean of two successive returns of +50% and −50% is 0%, but the geometric mean is −13% since \([(1 + 0.5)(1 - 0.5)]^2 = 0.87\) with \(d = 2\) periods. While some financial theory requires arithmetic mean as inputs (single-period Markowitz or mean-variance optimisation, single-period CAPM), most investors are interested in wealth compounding which is better captured by geometric means.

8.2 A few facts about moments

See books by Kendall et al. [1977], Vapnik [1998], and Kenney [2013], among others.

8.2.1 Defining the moments

We let \(X\) be an independent and identically distributed random variables. A central moment is a moment of the probability distribution of the random variable \(X\) around its mean. That is, the \(n\)th central moment of a real-valued random variable \(X\) is

\[ \mu_n - E[(X - \mu[X])^n], n > 1 \] (8.4)

where \(\mu_n - E[X^n]\) is the \(n\)th non-central moment (or raw moment). We define the cumulant of the random variable \(X\) as

\[ \kappa_n = \frac{d^n}{d\lambda^n} \ln E[e^{\lambda X}]_{\lambda = 0}, n = 1, 2, ... \] (8.5)

A fundamental task in many statistical analyses is to characterise the location and variability of a data set. The expectation is \(\mu_1 = \mu[X] = E[X]\), and we call the second central moment \(\mu_2\), the variance, denoted \(\sigma^2_X\). The variance of a random variable \(X\) is

\[ \sigma^2_X = Var(X) = E[X^2] - (E[X])^2 = \mu_2 - \kappa_2 \]

A further characterisation of the data includes skewness and kurtosis.

8.2.1.1 The skewness

The skewness of a random variable \(X\) is the third standardised moment given by

\[ S(X) = -E[\left(\frac{X - \mu_X}{\sigma_X}\right)^3] - \frac{\mu_3}{\sigma_3} - \frac{\kappa_3}{\kappa_2^{3/2}} \]

\[ = E[X^3] - 3\mu_X (E[X^2] - \mu_X E[X]) - \mu_3^3 \]

\[ = E[X^3] - 3\mu_X \sigma_X^2 - \mu_3^3 \]

where \(\mu_X\) is the mean and \(\sigma_X\) is the standard deviation of \(X\), \(\mu_3\) is the third central moment, and \(\kappa_k\) is the \(k\)th cumulant. Here, \(E[X^3]\) is the third non-central moment. It is sometimes referred to as Pearson’s moment coefficient of skewness, or simply the moment coefficient of skewness. Skewness is a measure of the lack of symmetry. The skewness for a normal distribution is zero, and any symmetric data should have a skewness near zero. Negative values for the skewness indicate data that are skewed left and positive values for the skewness indicate data that are skewed right. Note, Skewness can be infinite.
8.2.1.2 The kurtosis The kurtosis is a measure of whether the data are heavy-tailed or light-tailed relative to a normal distribution. That is, data sets with high kurtosis tend to have heavy tails, or outliers. Data sets with low kurtosis tend to have light tails, or lack of outliers. The histogram is an effective graphical technique for showing both the skewness and kurtosis of data set. The kurtosis is the fourth standardised moment, defined as

\[ K(X) = E\left( \frac{X - \mu_X}{\sigma_X} \right)^4 - \frac{\mu_4}{\sigma^4} \]

where \( \mu_4 \) is the fourth central moment and \( \sigma \) is the standard deviation. For any outlier \( x_p \), the value \( x_p - \mu \) tends to be large. Here, the outliers contribution to the sum is raised to the power of four. As a result, outliers significantly impact the value of the kurtosis. The kurtosis is bounded below by the squared skewness plus one:

\[ K(X) \geq (\frac{\mu_3}{\sigma})^2 + 1 \]

8.2.1.3 The general standardised summary statistics A standardised moment of a probability distribution is a moment (often a higher degree central moment) that is normalised, typically by a power of the standard deviation, rendering the moment scale invariant. The shape of different probability distributions can be compared using standardised moments. In the general case, we get the nth standardised moment as follows:

\[ \gamma_n(X) = \frac{\mu_n - \mu_{n-1}^2}{\sigma^n}, \quad n \geq 3 \]

which is the ratio of the nth moment about the mean, \( \mu_n \), to the nth power of the standard deviation

\[ \sigma^n = \mu_2 - \left( \sqrt{E[(X - \mu_X)^2]} \right)^n - \left( \sigma^2 \right)^{n/2} \]

Note, \( \gamma_1(X) = 0 \) and \( \gamma_2(X) = 1 \). As a result, we can compute the nth central moment as

\[ \mu_n(X) = \gamma_n(X)\sigma_n(X), \quad n \geq 3 \]

where \( \sigma_n(X) = (\sigma^2_X)^{n/2} \), and \( \mu_1 = \mu_X \) and \( \mu_2 = \sigma_X^2 \). We see that the third standardised moment is a measure of skewness, and the fourth standardised moment refers to the kurtosis.

8.2.2 The population mean and volatility

8.2.2.1 Description We let \( X_t \) be a Gaussian random variable with drift \( \mu \) and volatility \( \sigma \). We observe values uniformly allocated in time with time difference \( \delta \). That variable seen at time \( t + j\delta \) for the time \( t + 1 - (j + 1)\delta \) is given by

\[ X(j, j + 1) = \mu \delta - \sigma(W(j+1)\delta - W_j) \]

Given \( N \) period of time, and \( X_{j\delta} = X(j, j + 1) \), the first two sample moments are

\[ \hat{\mu}_N = \frac{1}{N} \sum_{j=0}^{N-1} X_{j\delta} \text{ and } \hat{\sigma}_N^2 = \frac{1}{N} \sum_{j=0}^{N-1} (X_{j\delta} - \hat{\mu}_N)^2 \]

(8.8)

Since the observations are selected randomly, both \( \hat{\mu}_N \) and \( \hat{\sigma}_N^2 \) are random variables. Their expected values can be evaluated by averaging over the ensemble of all possible samples of size \( N \) from the population. We get

\[ E[\hat{\sigma}_N^2] = \frac{n}{n-1} \sigma^2 \]
with population variance $\sigma^2 = E[X^2] - \mu^2$ where $\mu^2 = E[X] - E[X]E[X]$ due to independence of $X_i$ and $X_j$. Note, $\hat{\sigma}^2_N$ gives an estimate of the population variance that is biased by a factor of $\frac{n-1}{n}$. Hence, it is a biased sample variance. Correcting for this bias yields the unbiased sample variance

$$S^2 = \frac{n-1}{n} \hat{\sigma}^2_N$$

These two estimators are simply referred to as the sample variance.

### 8.2.2.2 Implementation

Given the sample mean $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$, we compute the sample mean and sample variance recursively as follows: Starting from the identities

$$(n + 1) \bar{X}_{n+1} - n \bar{X}_n + X_{n+1}$$

and

$$(n + 1) (\sigma^2_{n+1} + (\bar{X}_{n+1})^2) - n (\sigma^2_n + (\bar{X}_n)^2) + X_{n+1}^2$$

we get the recursive sample mean

$$\bar{X}_{n+1} = \bar{X}_n + \frac{X_{n+1} - \bar{X}_n}{n+1} \tag{8.9}$$

and the recursive sample variance

$$\sigma^2_{n+1} = \sigma^2_n + (\bar{X}_{n+1})^2 - (\bar{X}_{n+1})^2 + \frac{X_{n+1}^2 - \bar{X}_n^2}{n+1} \tag{8.10}$$

such that $\sigma^2_{n+1}$ only depends on $\sigma^2_n$, $\bar{X}_n$, $\bar{X}_{n+1}$ and $X_{n+1}$. If we normalise the data with a constant $K$ such that the ith value is $\frac{X_i}{K}$, we let $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} \frac{X_i}{K}$ be the sample mean and $\hat{\sigma}^2_n = \frac{1}{n} \sum_{i=1}^{n} (\frac{X_i}{K} - \bar{X})^2$ be the sample variance. Then, the recursive sample mean becomes

$$\tilde{X}_{n+1} = \tilde{X}_n + \frac{1}{n+1} (\frac{X_{n+1}}{K} - \tilde{X}_n) \tag{8.11}$$

and the recursive sample variance becomes

$$\tilde{\sigma}^2_{n+1} = \tilde{\sigma}^2_n + (\tilde{X}_{n+1})^2 - (\tilde{X}_{n+1})^2 + \frac{1}{n+1} (\frac{X_{n+1}^2}{K^2} - \tilde{\sigma}^2_n - (\tilde{X}_n)^2) \tag{8.12}$$
References


