A simple option price decomposition formula with applications to stochastic volatility calibration

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Abstract

For classical volatility models a decomposition formula for the option price is derived that expresses it as the sum of the price when correlation is zero plus a correlation correction term. The decomposition formula is subsequently expanded in the volatility-of-volatility parameter to yield expressions that are suitable for stochastic volatility model calibration.

1 A simple decomposition

Let us consider a general stochastic volatility (SV) model

\[
dS_t = \sigma_t S_t (\rho dW_t + \rho dZ_t), \quad d\sigma_t = a(\sigma_t) dt + b(\sigma_t) dW_t
\]

with \( W_t \) and \( Z_t \) standard Brownian motions, \( \rho \in (-1, 1) \) and \( \bar{\rho} := \sqrt{1 - \rho^2} \). Then the partial differential equation (PDE) satisfied by the call option

\[
C(S_t, K, \sigma_t) := E_t [(S_T - K) +]
\]

reads

\[
\frac{\partial C}{\partial t} + \mathcal{L}^0 C + \mathcal{L}^1 C = 0
\]

with

\[
\mathcal{L}^0 := a(\sigma_t) \frac{\partial}{\partial \sigma_t} + \frac{1}{2} b^2(\sigma_t) \frac{\partial^2}{\partial \sigma_t^2} + \frac{1}{2} \sigma_t^2 S_t^2 \frac{\partial}{\partial S_t^2}
\]

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and

\[ \mathcal{L}^1 := \rho b(\sigma_t) \sigma_t S_t \frac{\partial^2}{\partial S_t \partial \sigma_t} \]

(5)

We seek a solution of the form

\[ C(S_t, K, \sigma_t) = C^0(S_t, K, \sigma_t) + C^1(S_t, K, \sigma_t) \]

(6)

satisfying the boundary conditions

\[ C^0(S_T, K, \sigma_T) = (S_T - K)^+, \quad C^1(S_T, K, \sigma_T) = 0 \]

(7)

and where \( C^0(S_t, K, \sigma_t) \) satisfies

\[ \frac{\partial C^0}{\partial t} + \mathcal{L}^0 C^0 = 0 \]

(8)

By application of the Feynman-Kac Theorem the solution to the PDE (3) is therefore

\[ C(S_t, K, \sigma_t) = C^0(S_t, K, \sigma_t) + \rho E_t \left[ \int_t^T b(\sigma_u) \sigma_u S_u \frac{\partial^2 C^0(S_u, K, \sigma_u)}{\partial S_u \partial \sigma_u} \, du \right] \]

(9)

To rewrite the SV PDE solution in terms of Black-Scholes (BS) prices, recall the ‘mixing’ formula for \( \rho = 0 \) due to Hull and White [3]

\[ C^0(S_t, K, \sigma_t) = E_t \left[ C^{BS}(K, \sigma_{t,T}) \right] \]

(10)

where

\[ \sigma_{t,T} := \left( \frac{1}{T - t} \int_t^T \sigma_u^2 \, du \right)^{1/2} \]

(11)

Since we can find an implied volatility (IV) \( \Sigma^0_t \) such that \( C^{BS}(S_t, K, \Sigma^0_t) = C^0(S_t, K, \sigma_t) \) the mixing formula can be written as

\[ C^{BS}(S_t, K, \Sigma^0_t) = E_t \left[ C^{BS}(K, \sigma_{t,T}) \right] \]

(12)

Furthermore, for \( \rho \neq 0 \) it is also possible to find an IV \( \Sigma_t \) such that \( C^{BS}(S_t, K, \Sigma_t) = C(S_t, K, \sigma_t) \). Thus the decomposition formula (9) reads

\[ C^{BS}(S_t, K, \Sigma_t) = C^{BS}(S_t, K, \Sigma^0_t) + \rho E_t \left[ \int_t^T b(\sigma_u) \sigma_u S_u \frac{\partial^2 C^{BS}(S_u, K, \Sigma^0_u)}{\partial S_u \partial \Sigma^0_u} \frac{\partial \Sigma^0_u}{\partial \sigma_u} \, du \right] \]

(13)

Notice that equation (13) is an exact expression and that it is reminiscent of the decomp-
sition formula due to Alòs [1].

2 Expanding the decomposition

In this note the main application of the decomposition (13) is to the calibration of SV models. Assuming the necessary technical conditions are satisfied, expand the instantaneous volatility around some small parameter $\varepsilon$:

$$\sigma_t(\varepsilon) = \sigma_t(0) + \varepsilon \sigma'_t(0) + \frac{\varepsilon^2}{2} \sigma''_t(0) + \cdots$$

(14)

This leads to an expansion for the realized volatility $\sigma_{t,T}$

$$\sigma_{t,T}(\varepsilon) = \sigma_{t,T}(0) + \varepsilon \sigma'_{t,T}(0) + \frac{\varepsilon^2}{2} \sigma''_{t,T}(0) + \cdots$$

(15)

Next, an expansion for $\Sigma^0_t$ is found by making use of the mixing formula (12). This is achieved by expanding both sides of (12). Thus,

$$C^{BS}(S_t, K, \Sigma^0_t(\varepsilon)) = C^{BS}(S_t, K, \Sigma^0_t(0) + \varepsilon \Sigma'_t(0) + \frac{\varepsilon^2}{2} \Sigma''_t(0) + \cdots)$$

$$= C^{BS}(S_t, K, \Sigma^0_t(0)) + \left\{ \varepsilon \Sigma'_t(0) + \frac{\varepsilon^2}{2} \Sigma''_t(0) \right\} v^{BS}(S_t, K, \Sigma^0_t(0))$$

$$+ \frac{1}{2} \left(\varepsilon \Sigma'_t(0)\right)^2 v^{BS}(S_t, K, \Sigma^0_t(0)) + O(\varepsilon^3)$$

(16)

and

$$E_t [C^{BS}(K, \sigma_{t,T})] = E_t \left[ C^{BS}(S_t, K, \sigma_{t,T}(0) + \varepsilon \sigma'_{t,T}(0) + \frac{\varepsilon^2}{2} \sigma''_{t,T}(0) + \cdots) \right]$$

$$= E_t \left[ C^{BS}(S_t, K, \sigma_{t,T}(0)) \right]$$

$$+ E_t \left\{ \varepsilon \sigma'_{t,T}(0) + \frac{\varepsilon^2}{2} \sigma''_{t,T}(0) \right\} v^{BS}(S_t, K, \sigma_{t,T}(0))$$

$$+ E_t \left[ \frac{1}{2} \left(\varepsilon \sigma'_{t,T}(0)\right)^2 v^{BS}(S_t, K, \sigma_{t,T}(0)) \right] + O(\varepsilon^3)$$

(17)

In the examples we will consider $\varepsilon$ is in fact the volatility of volatility (henceforth ‘vol-of-vol’) parameter. The value $\varepsilon = 0$ therefore corresponds to deterministic volatility. In other words $\sigma_t(0)$ is deterministic and thus also $\sigma_{t,T}(0)$. With this in mind, equating powers of $\varepsilon$
gives the following identifications:

\[
\Sigma^0_t(0) = \sigma_{t,T}(0) \\
\Sigma^{0'}_t(0) = E_t [\sigma'_{t,T}(0)] \\
\Sigma^{0''}_t(0) = E_t \left[ \sigma''_{t,T}(0) \right] + \frac{v_0 BS(S_t, K, \sigma_{t,T}(0))}{v_0 BS(S_t, K, \sigma_{t,T}(0))} \left\{ E_t \left[ (\sigma'_{t,T}(0))^2 \right] - (E_t [\sigma'_{t,T}(0)])^2 \right\}
\]

which specifies the expansion

\[
\Sigma_t(\varepsilon) = \sigma_{t,T}(0) + \varepsilon E_t [\sigma'_{t,T}(0)] + \frac{\varepsilon^2}{2} E_t \left[ \sigma''_{t,T}(0) \right] + O(\varepsilon^3)
\]

In a similar manner \( S_u = S_u(\varepsilon) \) since

\[
\log S_u(\varepsilon)/S_t = -\frac{1}{2} \int_t^u \sigma_r^2(\varepsilon) \, dr + \int_t^u \sigma_r(\varepsilon) (\rho dW_r + \bar{\rho} dZ_r)
\]

Hence we may also write

\[
S_u(\varepsilon) = S_u(0) + \varepsilon S'_u(0) + \frac{\varepsilon^2}{2} S''_u(0) + O(\varepsilon^3)
\]

Given the expansions (14) and (15) this can be calculated.

If we look at the decomposition (13) we see that we have the expansions for the right hand side of the equation. The last step is to write

\[
\Sigma_t(\varepsilon) = \Sigma_t(0) + \varepsilon \Sigma'_t(0) + \frac{\varepsilon^2}{2} \Sigma''_t(0) + O(\varepsilon^3)
\]

and expand the left hand side of the decomposition formula. Equation powers of \( \varepsilon \) then leads to identifications for \( \Sigma_t(0), \Sigma'_t(0) \) and \( \Sigma''_t(0) \).

The ideas just discussed is best illustrated through an example, which is the topic of the next section.

### 3 Examples

In this section we will carry out an expansion to first order in the parameter \( \varepsilon \) under the SABR [2] model. Recall that it reads

\[
d\sigma_t(\varepsilon) = \varepsilon \sigma_t(\varepsilon) dW_t
\]
Here $\sigma_t$ denotes the instantaneous volatility, the constant $\alpha$ is the (SABR) volatility of volatility (henceforth vol-of-vol), and $\varepsilon$ a dimensionless parameter around which the expansion will be carried out. Effectively $\varepsilon$ is the vol-of-vol parameter as we can always write $\tilde{\alpha} := \varepsilon \alpha$.

From (22) it follows that

$$\sigma_u^2(\varepsilon) = \sigma_t^2 e^{-\varepsilon^2 \alpha^2 (u-t) + 2\varepsilon \alpha W_{t,u}}$$

(23)

where we have introduced the notation $W_{t,u} := W_u - W_t$. We can expand $\sigma_u^2(\varepsilon)$ around $\varepsilon = 0$:

$$\sigma_u^2(\varepsilon) = \sigma_u^2(0) + \varepsilon \frac{d\sigma_u^2}{d\varepsilon}(0) + \frac{\varepsilon^2}{2!} \frac{d^2\sigma_u^2}{d\varepsilon^2}(0) + \cdots$$

(24)

This gives us (where the explicit dependence of $\sigma_u^2$ on $\varepsilon$ will from now on be suppressed)

$$\sigma_u^2 = \sigma_t^2 \left\{ 1 + 2\varepsilon \alpha W_{t,u} + \varepsilon^2 \alpha^2 (2W_{t,u}^2 - (u-t)) + \cdots \right\}$$

(25)

Next, let us expand $\sigma_{t,T}$:

$$\left( \frac{1}{T-t} \int_t^T \sigma_u^2 \, du \right)^{1/2} = \sigma_t + \frac{1}{2} \frac{\sigma_t}{T-t} \int_t^T \left\{ 2\varepsilon \alpha W_{t,u} + \varepsilon^2 \alpha^2 (2W_{t,u}^2 - (u-t)) \right\} du$$

$$- \frac{1}{8} \frac{\sigma_t}{(T-t)^2} \left( \int_t^T 2\varepsilon \alpha W_{t,u} \, du \right)^2$$

(26)

Taking expectations yields

$$E_t \left[ \left( \frac{1}{T-t} \int_t^T \sigma_u^2 \, du \right)^{1/2} \right] = \sigma_t \left( 1 + \frac{\varepsilon^2 \alpha^2 (T-t)}{12} \right) + O(\varepsilon^3)$$

(27)

Comparing this to the expression (18), (and remembering that we are doing a first order expansion) we see that

$$\Sigma_t^0 = \sigma_t + O(\varepsilon^2)$$

(28)

and therefore

$$\frac{\partial \Sigma_t^0}{\partial \sigma_t} = 1 + O(\varepsilon^2)$$

(29)

Because in the SABR model $b(\sigma_t) = \varepsilon \alpha \sigma_t$ the decomposition formula reads

$$C^{BS}(S_t, K_t, \Sigma_t) = C^{BS}(S_t, K_t, \sigma_t) + \varepsilon \rho \alpha E_t \left[ \int_t^T \sigma_u^2 S_u \frac{\partial^2 C^{BS}(S_u, K, \Sigma_u)}{\partial S_u \partial \Sigma_u} \, du \right]$$

(30)
As the second term on the right hand side is already multiplied with $\varepsilon$, in a first order expansion we can set $\sigma_u = \sigma_t$. This also means that $S_u$ can be taken to evolve with a constant volatility $\sigma_t$ and that the expectation is the expectation under the BS model with constant volatility $\sigma_t$. This leads us to

$$C^{BS}(S_t, K, \Sigma_t) = C^{BS}(S_t, K, \sigma_t) + \varepsilon \rho \sigma_t^2 E^{BS}_t \left[ \int_t^T S_u \nu^{BS}_u(S_u, K, \sigma_t) \, du \right]$$

(31)

where $\nu^{BS}$ is the BS vanna.

Lastly, by making use of $\Sigma_t = \Sigma_t(0) + \varepsilon \Sigma_t'(0) + \cdots$, expanding and subsequently equating powers of $\varepsilon$ gives

$$\Sigma_t(0) = \sigma_t, \quad \Sigma_t'(0) = \frac{\rho \sigma_t^2}{\nu^{BS}(S_t, K, \sigma_t)} E^{BS}_t \left[ \int_t^T S_u \nu^{BS}_u(S_u, K, \sigma_t) \, du \right]$$

(32)

with $\nu^{BS}$ the BS vega. Thus, to first order the implied volatility in the SABR model is

$$\Sigma_t = \sigma_t + \frac{\varepsilon \rho \sigma_t^2}{\nu^{BS}(S_t, K, \sigma_t)} E^{BS}_t \left[ \int_t^T S_u \nu^{BS}_u(S_u, K, \sigma_t) \, du \right] + O(\varepsilon^2)$$

(33)

The expectation of the integral can be calculated very easily as it is under the BS model. In fact it can be evaluated analytically.

Although only a first order expansion in $\varepsilon$ has been carried out, it is clear (albeit potentially tedious) how to include higher order terms. The result of the expansion will be terms that can be evaluated numerically or analytically under a BS world with constant volatility. The numerical evaluation will still be faster than a numerical evaluation under the ‘SV measure’ as dimensionality has been reduced by one.

References

