On the Spectral Decomposition of Portfolio Skewness and its Application to Portfolio Optimization

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Abstract

This work presents a new convex risk measure that we call negative quadratic skewness that is an approximation of the negative component of portfolio skewness. This risk measure allows us to increase portfolio skewness through the minimization of the negative quadratic skewness. First, we show how to split portfolio skewness using spectral decomposition of the block diagonal coskewness matrix as the sum of two components: spectral positive and negative skewness. Then, we define the quadratic skewness as the difference of two convex positive semidefinite quadratic forms and show that it is not possible to maximize or minimize it because is nonconvex. Then, we propose an heuristic to increase skewness through the minimization of the negative component of quadratic skewness. Finally, we posed a portfolio optimization model, that is similar to Markowitz model, and allows us to minimize the negative quadratic skewness of portfolios using quadratic programming or second order cone programming.

Keywords: finance, investments, convex programming, portfolio optimization, skewness.

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1 Introduction

Higher order moment portfolio optimization is a topic that has been studied in last three decades, mean variance skewness model was studied in Konno and ichi Suzuki (1995) and Konno and Yamamoto (2005), where they proposed a piecewise linear approximation of skewness using a mixed integer linear programming; and Lai (1991) proposed a formulation assuming that variance is one. The mean skewness kurtosis model was studied in Athayde and Flôres (2003). In the case of mean kurtosis model, Cajas (2023c) proposed a convex formulation of kurtosis and Cajas (2023a) proposed an approximation of kurtosis that increase the speed of the calculation. If we talk about four moments portfolio, there are several algorithms that allow us to solve this problem due to complexity that arise from nonconvexity of third and fourth moments like Jurczenko et al. (2015), Niu et al. (2019) and Zhou and Palomar (2020), however they are very complex to implement. Also, Cajas (2023b) proposed an alternative way to incorporate skewness and kurtosis through linear moments or L-moments.

In this work we propose a new risk measure that we call negative quadratic skewness that allow us to increase portfolio skewness minimizing an approximation of the negative component of skewness. The minimization of negative quadratic skewness can be posed as a quadratic problem and as a second order cone constraint using the disciplined convex programming (DCP) methodology\(^1\) in the same way like Markowitz (1952) model, making it suitable for large scale problems. Also, this model allow us to incorporate the negative quadratic skewness in problems like minimization of risk, maximization of risk averse utility function, maximization of return risk ratio and risk parity based on negative quadratic skewness.

2 Spectral Decomposition of Portfolio Skewness

We can express portfolio skewness:

\(^1\)Grant and Boyd (2006)
\[ \sigma_3^2(x) = x'M_3(x \otimes x) \]
\[ M_3 = [S_1 | S_2 | \ldots | S_n] \]
\[ S_i = \begin{bmatrix}
  s_{i11} & s_{i12} & \ldots & s_{i1n} \\
  s_{i21} & s_{i22} & \ldots & s_{i2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  s_{in1} & s_{in2} & \ldots & s_{inn}
\end{bmatrix} \]

where \( M_3 \) is the coskewness tensor, \( S_i \) are the faces of coskewness tensor that are symmetric by construction, \( x \) is portfolio weights vector and \( \otimes \) is the kronecker product. We can transform the formula above using the block diagonalization operator block\_diag^n(\cdot) over the matrixes \( S_i \) and the ones column vector \( 1_n \) of size \( n \) as follows:

\[ \sigma_3^2(x) = (1_n \otimes x)'\Sigma_3(x \otimes x) \]
\[ \Sigma_3 = \text{block\_diag}^n(M_3) \]
\[ \Sigma_3 = \begin{bmatrix}
  S_1 & 0 & \ldots & 0 \\
  0 & S_2 & \ldots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \ldots & S_n
\end{bmatrix} \]
\[ 1_n = \begin{bmatrix} 1 & 1 & \ldots & 1 \end{bmatrix}' \]

If we develop the expression above we get:

\[ \sigma_3^2(x) = (1_n \otimes x)'\Sigma_3(x \otimes x) \]
\[ \sigma_3^2(x) = \begin{bmatrix} x' & x' & \ldots & x' \end{bmatrix} \Sigma_3 \begin{bmatrix} x_1x & x_2x & \ldots & x_nx \end{bmatrix} \]
\[ \sigma_3^2(x) = x_1(x'S_1x) + x_2(x'S_2x) + \ldots + x_n(x'S_nx) \]
\[ \sigma_3^2(x) = \sum_{i=1}^{n} x'(x_iS_i)x \]

Then, if we split the matrices \( S_i \) using spectral decomposition as the sum of a positive semidefinite \( S_i^+ \) and a negative semidefinite matrix \( S_i^- \), this means that \( S_i = S_i^+ + S_i^- \), we can split portfolio skewness into two components:
\[ \sigma_3^\alpha(x) = \sum_{i=1}^{n} x'(x_i S_i^+) x - \sum_{i=1}^{n} x'(-x_i S_i^-) x \]
\[ \sigma_3^\beta(x) = \left( \sum_{i=1}^{n} x_i S_i^+ \right) x - \left( \sum_{i=1}^{n} -x_i S_i^- \right) x \]

Positive Spectral Skewness  
Negative Spectral Skewness

We call the first component as positive spectral skewness and the second negative spectral skewness. In the case of long only portfolios, we have a portfolio with positive (negative) skewness when the positive spectral skewness is higher (lower) than negative spectral skewness. In the case of long short portfolios, there are cases when the spectral positive (negative) skewness is negative (positive) and the total skewness of portfolio is negative (positive). We can notice that this behavior is due the \( x_i \) variables that multiply each \( S_i^+ \) and \( S_i^- \) matrix, also if we remove these variables \( x_i \) from both components of skewness they are transformed into quadratic forms. Using this idea we propose an alternative measure of portfolio skewness removing the variables \( x_i \) from both components of portfolio skewness:

\[ \nu_3^\alpha(x) = x' \left( \sum_{i=1}^{n} S_i^+ \right) x - x' \left( \sum_{i=1}^{n} -S_i^- \right) x \]
\[ \nu_3^\beta(x) = x' V^+ x - x' V^- x \]
\[ \nu_3^\gamma(x) = \frac{\nu_3^{\alpha+}(x)}{\nu_3^{\alpha-}(x)} - \frac{\nu_3^{\beta+}(x)}{\nu_3^{\beta-}(x)} \]

Positive Quadratic Skewness  
Negative Quadratic Skewness

where the matrices \( V^+ \) and \( V^- \) are positive semidefinite. This new measure of skewness, that we call quadratic skewness \( \nu_3^2 \), is expressed as the difference of two positive convex expressions: the first represents the positive component of skewness and the second is the negative component of skewness. Due that in practice \( \nu_3^2 \) is the sum of a convex \( (\nu_3^{2+}(x)) \) and concave \( (-\nu_3^{2-}(x)) \) expressions, we can not maximize or minimize \( \nu_3^2 \). However, one approach to increase the quadratic skewness is to minimize \( \nu_3^{2-}(x) \), an idea that we develop in the following section.

3 Minimization of Quadratic Negative Skewness

The negative component of quadratic skewness \( \nu_3^{2-}(x) = x' V^- x \) is convex and positive, so one approach to reduce the impact of this component on quadratic skewness is to minimize the negative quadratic skewness (approximate to zero). If we posed this idea
as an optimization problem we get the following problem:

\[
\min_x \sum_{i=1}^{n} x'V^-x
\]

\[
\text{s.t.} \sum_{i=1}^{n} x_i = 1
\]

\[
x \geq 0
\]

(1)

The model above is a quadratic problem similar to Markowitz model. However, taking advantage of second order cone we can represent the minimization of negative quadratic skewness as follows:

\[
\min_{x, \nu} \nu
\]

\[
\text{s.t.} \|V^{1/2}x\| \leq \nu
\]

\[
\sum_{i=1}^{n} x_i = 1
\]

\[
x \geq 0
\]

(2)

The model above allow us to minimize the square root of negative quadratic skewness. This formulation is very practical because allow us to increase the skewness of portfolio solving a second order cone problem similar to the minimization of standard deviation. Also, we can use this formulation in combination with other convex risk measures in order to add positive skewness to our custom portfolios.

4 Numerical Examples

We select 30 assets (i.e., stocks JCI, TGT, CMCSA, CPB, MO, T, APA, MMC, JPM, ZION, PSA, BAX, BMY, AAPL, PCAR, BA, TMO, TXT, DE, MSFT, HPQ, SEE, VZ, CNP, NI, JNJ, PFE, AMZN, GE and GOOG) from the S&P 500 (NYSE) and download daily adjusted closed prices from Yahoo Finance for the period from January 1, 2016 to December 30, 2021. Then, we calculated daily returns building a returns matrix of size \( T = 1508 \) and \( N = 30 \). To calculate the portfolios we use Python 3.9, CVXPY and MOSEK solver.
4.1 Minimization of Quadratic Negative Skewness

To show how this new risk measure adds positive skewness to other risk measures we are going to compare three portfolios: minimum standard deviation ($\sigma(x)$), minimum square root quadratic negative skewness ($\nu^{-3}(x)$), and minimization of a combined risk measure of standard deviation and square root negative quadratic skewness ($\sigma(x) + 8\nu^{-3}(x)$).

![Figure 1: Optimal Portfolios in Mean Standard Deviation Plane](image)

In figure 1 we can see that in the mean standard deviation plane, the minimum square root negative quadratic skewness and the minimum combined risk measure are inside the efficient frontier. Also, we can notice that the skewness of portfolios are distributed in a messy way along the mean standard deviation plane.
In figure 2 we can see that in the mean skewness plane, the minimum square root quadratic negative skewness and the minimum combined risk measure have a higher skewness than minimum standard deviation portfolios. This means that the addition of the square root negative quadratic skewness in the objective function increase the skewness of the minimum standard deviation portfolio.

5 Conclusions

The negative quadratic skewness is a new risk measure that allow us to approximate the negative component of portfolio skewness and gives an alternative way to add positive skewness to our portfolios. The main advantage is that this portfolio optimization model can be expressed as quadratic form or using a second order cone constraint, making this formulation suitable for large scale problems. The resulting model is very similar to Markowitz but instead of using covariance matrix we use the sum of faces of coskewness tensor. Another advantage is that this formulation can be solved using state of art solvers that supports quadratic programming or second order cone programming. Finally, this formulation is very flexible because can be combined with other risk measures or used to pose other portfolio problems like risk constraints, maximization of return risk ratio or risk parity.
References


