Implied Impermanent Loss: A Cross-Sectional Analysis of Decentralized Liquidity Pools

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30th April 2024

Abstract

We propose a continuous-time stochastic model to analyze the dynamics of the impermanent loss, as the variance of the relative price on the underlying tokens, in decentralized liquidity provision. We estimate the risk-neutral joint distribution of the tokens by minimizing the Hansen–Jagannathan bound, which we then use to value the impermanent loss and for calculating an implied correlation of the token pair. We explore how implied volatilities and correlations affect impermanent loss, revealing their role in explaining the cross-sectional returns of liquidity pools.

We test our hypothesis on options data from a major centralized derivative exchange.

Keywords: Decentralized Exchanges, Decentralized Finance, Impermanent Loss, Derivatives, Risk-Neutral Pricing, Staking, Yield Farming.

JEL Classification Codes: G10, G11, G13, G20

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We received helpful comments and suggestions from Faycal Drissi, Roman Kozhan, Alfred Lehar, Elisa Luciano, Roberto Marfe, Giovanna Nicodano, Kirill Shakhnov (discussant), Claudio Tebaldi, and Fabio Trojani. We thank Mariia Aksenova and Roman Lewandrowski for their help accessing on-chain data. We also thank the participants of the Decentralised Finance Research Group Oxford-Man Institute (DeFOx), the Operations Research Seminar at North Carolina State University, the LTI Research Seminar at the Collegio Carlo Alberto, and the ToDeFi 2024. In addition, we thank the FinTech & Digital Finance Chair of Université Paris Dauphine-PSL in partnership with Mazars and Crédit Agricole CIB, and the Avalanche Foundation for financial support.

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Electronic copy available at: https://ssrn.com/abstract=4811111
1 Introduction

This article focuses on the risk and returns of decentralized liquidity provision (often referred as to yield farming), which is the practice of depositing tokens into a liquidity pool on a decentralized exchange (DEX) (Lehar and Parlour (2023), Park (2023), Capponi and Jia (2024), Heimbach et al. (2022)). A liquidity taker uses the pool to buy and sell tokens by swapping them according to a price given by an automated market maker (AMM) using a transparent algorithmic pricing rule. In exchange for providing liquidity, the liquidity provider (LP) earns a share of the trading fees paid by liquidity takers (traders) who utilize the pool. These rewards can be substantial but come with considerable risks, especially due to the so-called impermanent loss which is the adverse selection faced by LPs (Milionis et al. (2023), Li et al. (2023)).

The LP faces impermanent loss whenever the ratio, i.e., the relative price of the tokens fluctuates. Because liquidity was provided at a specific ratio of the two assets, if the price of one asset increases or decreases significantly compared to the other, the AMM will adjust the ratio in the pool to reflect the new prices. As a result, the pool tokens still represent the same proportion of the pool, but the value of the assets originally deposited may have changed. The relative price of the tokens fluctuates due to innovations in the price of one of the two tokens or if there’s a divergence in token prices, for example, if the tokens are negatively correlated. Hence, the major risks for LPs stems from the tokens’ volatility and their time-varying correlations.

Option prices, by construction, reflect investors’ expectations about future price movements and their moments. The main objective of this paper is to derive the risk-neutral valuation of impermanent loss and to study the economics and the channels through which it is related to future investment opportunities. We show that the implied impermanent loss is composed of its three (implied) key determinants—the two tokens’ idiosyncratic risk, and their diversification risk. In the next step we investigate the predictive power of the (option) implied impermanent loss which explains the cross-section of returns on liquidity pools for liquidity providers. Our paper therefore integrates two different markets, the market for decentralized liquidity provision and the centralized derivatives market. We now discuss these findings in more detail.

The construction of our model starts with a mathematical formulation with an AMM that governs prices with a constant product rule. We derive and characterize the impermanent loss as the major

1Example: Consider two tokens, A and B, each initially worth 1 USD. The AMM holds identical quantities of 100 of each token and offers both at a fixed exchange rate of 1:1. Suppose token B’s price appreciates to 2 USD in the wider market. Arbitrageurs exchange all of token B in the pool for token A because token B is more valuable. The pool then holds 200 of token A worth 200 USD. If, however, the LP did not participate in liquidity provision and held only A and B, his portfolio value would be 300 USD. Therefore, the impermanent loss equals 100 USD (the difference between 300 USD and 200 USD).
source of risk; our calculations show that the impermanent loss is a function of the respective token return volatilities and the correlation between them (which we often refer to as the “drivers” or “building blocks” of the impermanent loss).

In the next step, we explore the valuation of the implied impermanent loss from a risk-neutral perspective. Impermanent loss is essentially one-eighth of the realized variance of the relative price and has a valuation equal to a risk-neutral expectation of the log contract, which can obtained through a portfolio of European options on the relative price (Carr and Madan (1999)). Nevertheless, two primary challenges arise in conducting this model-free valuation for the relative price of the token pairs: First, the options market for the relative price is non-existent. Instead, liquid options are traded only directly on individual tokens. Second, the relative price is not a martingale. To address the first difficulty, we compute a joint density derived from existing options on separate tokens. To tackle the second challenge, we will utilize a change of numéraire.

To estimate an implied multivariate distribution accurately, we rely on a method that adheres to the principles of arbitrage theory. This approach ensures that market prices remain consistent with observations and avoids creating arbitrage opportunities. A widely accepted technique for achieving this is minimizing the Hansen and Jagannathan (1991) (HJ) bound. The HJ bound sets an upper limit on how much return a portfolio can generate relative to its risk. This optimization task can be solved using quadratic programming techniques. To better understand joint-token risk, we analyze implied correlations using spread options and the Margrabe formula. Implementation of the Margrabe formula involves choosing one of the tokens to be the numéraire under a new risk-neutral measure, and under this measure the relative price becomes a martingale, thus allowing us to utilize the formula of Carr and Madan (1999) to value the variance swap. In addition, the risk-neutral implied multivariate distribution allows us to construct an option implied correlation even without the existence of an implied volatility originating from an index of cryptocurrencies.\(^2\) Prior studies (e.g., Longin and Solnik (2001), Ang and Chen (2002), and Driessen et al. (2009)) suggest that the average equity market correlation is highly informative and either measures systematic risk in financial markets, diversification benefits, or tail risk. Our measure of correlation transfers this concept to the cryptocurrency market.

Our empirical analysis begins by presenting stylized facts on returns and risks (impermanent loss) in liquidity provision. We examine a large cross-section of Uniswap liquidity pools, the leading decentralized exchange on Ethereum. Our results show Uniswap pools yield an average APR of 15%, peaking near 50%. We note substantial impermanent losses, averaging -10% and reaching lows of -70%.

For equities Elton and Gruber (1973) are one of the first to use this average correlation using realized returns. Driessen et al. (2005) and Skinzi and Refenes (2005) introduced it under the risk-neutral measure, with the later literature (Driessen et al. (2012), Buss et al. (2017), Schoeneleber (2023)) referring to it as equicorrelation.
Next, we concentrate on deriving implied impermanent loss and its building blocks for the BTC-ETH token pair from options data, given the liquid cryptocurrency derivatives market for Bitcoin (BTC) and Ethereum (ETH). While the implied impermanent loss and the implied correlation rely on the implied multivariate distribution, the idiosyncratic tokens’ risks can be inferred from the (Model-Free) Implied Variance (IV) like the volatility index (VIX) (Britten-Jones and Neuberger (2000), Carr and Madan (1999), Bakshi et al. (2015), and Martin (2016)). Consistent with our framework, we find that the implied token variance of ETH and the implied correlation between BTC and ETH explains more than 60% (and up to 90%) of the daily levels in the implied impermanent loss. Intuitively, shocks to one underlying increase implied token variance, consequently raising impermanent loss. Conversely, higher implied correlation lowers impermanent loss as positive token comovement does not impact relative prices. In the last step, we compare our implied variables to their realized counterparts constructed from historical data. For example, the implied impermanent loss is on average at 0.14 while the realized counterpart is at 0.09.

In the next step, we explore the implied impermanent loss and its components’ sensitivity to abrupt changes in the underlying assets by examining implied measures during events with the most significant simultaneous drawdowns in both BTC and ETH. We observe a sharp rise especially in implied volatility, reflecting a leverage effect. We examine the implied correlation dynamics by analyzing days when one token experiences significant declines while the other remains stable. As anticipated, there is a notable decrease in implied correlation, concurrent with an increase in implied impermanent loss.

We study the cross-section of returns for liquidity provision to examine the existence of a risk-return relationship in liquidity pool returns. Our analysis shows impermanent loss as a key risk factor linking risks and returns in liquidity provision: Suppose an LP deposits assets into a liquidity pool. An innovation in the price of one token leads to a change in the relative price and prompts arbitrageurs to exploit price changes, causing impermanent loss for the LP. However, this temporary setback is met with increased trading activity within the pool. As a result, the increased trading volume, leads to higher APR, ultimately rewarding the LP’s participation in liquidity provision. We therefore uncover a positive link between innovation ("shocks") in fundamentals measured by implied quantities, the impermanent loss, and returns from liquidity provision.

The results from the Fama and MacBeth (1973) two-stage cross-sectional regressions confirm our reasoning, displaying a positive significant price of risk of the realized impermanent loss for short future horizons like the next one or two days. Motivated by equity factor studies, we explore the significance of our implied variables—specifically, the “expectation of moments of price innovation” reflected in the implied impermanent loss, alongside its drivers (individual implied volatilities and
implied correlation) in the cross-section of liquidity pools for providers. Unlike realized impermanent loss, implied impermanent loss explains the cross-section even for horizons up to 7 days. The results are qualitatively similar considering the drivers as risk factors. While the price of risk for the individual implied volatilities is positive, in liquidity provision, LPs favor high positive correlations among tokens to mitigate impermanent loss. Consequently, the price of risk for the implied correlation is negative. Our results are consistent when running the cross-sectional procedure in a multivariate setting where the drivers jointly boost the significance to 10 days with consistent signs.

The rest of this article is organized as follows: Section 2 discusses the literature. Section 3 formulates the mathematical models. Section 4 replicates the variance swap representing the implied impermanent loss. Section 5 describes the preparation of the data. Section 6 demonstrates the empirical applications. Section 7 provides robustness tests, and Section 8 concludes.

2 Literature Review

Cousaert et al. (2022) study the general framework of yield farming by focusing on the protocols and tokens used by aggregators. Heimbach et al. (2022) analyze in detail the risks and returns of Uniswap V3 liquidity providers. A continuous-time framework of yield farming from the view of the yield farmer is developed in Li et al. (2023). Milionis et al. (2023) have identified loss-versus-rebalancing as the primary risk for DeFi liquidity providers within the framework of prices following geometric Brownian motion. They further decompose this risk into an adverse selection cost and an information cost. Cartea et al. (2023) introduce a new comprehensive metric of predictable loss for liquidity providers and derive an optimal liquidity provision strategy. Augustin et al. (2022) study LP token staking and the return chasing behavior of investors on PancakeSwap. Lehar and Parlour (2023) show that the equilibrium size of a pool balances the fee revenue against the impermanent loss.

We contribute to this literature by developing a continuous-time mathematical framework replicating impermanent loss under the risk-neutral measure and establishing a risk-return relationship explaining cross-sectional returns in liquidity provision.

Options prices, by construction, reflect investors’ expectations about future price movements. The literature on option-implied information for equity is extensive (see Christoffersen et al. (2011) for an overview). In contrast, the literature on option-implied volatility for cryptocurrencies is still developing. Alexander and Imeraj (2021) construct a term structure of bitcoin implied volatility indices covering maturities from one week to three months. We contribute to this body of literature by developing a new
methodology that allows us to apply established metrics from the equity market to the cryptocurrency markets, such as option-implied correlation, and to quantify it empirically.

The literature concerning replicating impermanent loss is limited, with only a few exceptions. Mas-aaki Fukasawa and Wunsch (2023) investigate the link between Constant Function Markets, variance swaps, and gamma swaps, and replicate the impermanent loss with a weighted variance swap whenever the numeraire of the token pair is a stablecoin. A similar assumption is made by Clark (2020a) and Clark (2020b) who argues that participating in liquidity provision and encountering impermanent loss essentially involves taking a short volatility position. We contribute to this literature by developing a methodology for replicating impermanent loss, applicable even when both tokens are non-stable coins, and empirically quantifying the implied impermanent loss for the BTC-ETH token pair. Our paper therefore fills the gap in the literature and connects more broadly the literature on implied volatility, implied correlation, and implied impermanent loss to liquidity provision in decentralized finance via the derivatives market.

3 Mathematical Formulation

In this section, we formulate the mathematical model of an LP. We start by assuming a pool with a constant product AMM rule\(^3\) and assuming that the underlying token prices follow a geometric Brownian motion. These assumptions lead to a characterization of the impermanent loss in terms of the volatility of the tokens’ relative price.

3.1 Dynamics Under a Constant Product Rule

Let \(N_1(t)\) and \(N_2(t)\) denote the amounts of two respective tokens that are paired for trading in a liquidity pool. The constant product rule for this liquidity pool is as follows:

\[
L = \sqrt{N_1(t)N_2(t)},
\]

where \(L > 0\) is a constant. From constant product rule (3.1) a relative price of token emerges,

\[
R(t) = \frac{N_2(t)}{N_1(t)} = \text{token 2 per token 1},
\]

\(^3\)For the sake of simplicity, we do not consider pools with concentrated liquidity (CL), a concept that is introduced by Uniswap V3, enabling LPs to consolidate their pool liquidity within a defined range and earn fees when the spot price enters their specified active zone. For a detailed exploration of concentrated liquidity, refer to Heimbach et al. (2022).
from which we can deduce the amount of each token in the liquidity pool,

\[ N_1(t) = \frac{L}{\sqrt{R(t)}}, \quad N_2(t) = L\sqrt{R(t)}. \]  

(3.3)

We make the following assumption about the market prices and the relative price for tokens in the liquidity pool.

**Assumption 3.1.** The relative token price in a liquidity pool is equal to the ratio of the market prices of tokens,

\[ R(t) = \frac{P_1(t)}{P_2(t)} = \text{dollar per token 1} \text{dollar per token 2} = \text{token 2 per token 1}, \]  

(3.4)

where \( P_i(t) \) for \( i = 1, 2 \) are the market prices of tokens outside of the liquidity pool.

The rationale for equation (3.4) is that if it were not true, then arbitrageurs would enter the liquidity pool and exploit the price discrepancy until it corrected itself. Thus, arbitrageurs will ensure that the ratio of \( N_1(t) \) to \( N_2(t) \) in the pool maintains a relative price equal to the relative price of the greater market external of the pool.

Our model takes token prices to be given by geometric Brownian motions (GBMs),

\[ dP_i(t) = \mu_i P_i(t) dt + \sigma_i(t) P_i(t) dB_i(t), \quad i = 1, 2, \]  

(3.5)

where \( B_i(t) \) for \( i = 1, 2 \) are two correlated standard Brownian motions defined on a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\), \( dB_1(t) dB_2(t) = \rho dt, \rho \in [-1, 1] \) is the correlation between them, and \( \sigma_i > 0 \).

We apply the Itô’s formula to the ratio of \( P_1(t) \) and \( P_2(t) \) to get the stochastic differential equation for the relative price \( R(t) \),

\[ dR(t) = \tilde{\mu}(t) R(t) dt + \tilde{\sigma}(t) R(t) dB(t), \]  

(3.6)

where

\[ \tilde{\mu}(t) = \mu_1 - \mu_2 + \sigma_2^2(t) - \sigma_1(t) \rho \]  

(3.7)

\[ \tilde{\sigma}^2(t) = \sigma_1^2(t) - 2\rho \sigma_1(t) \sigma_2(t) + \sigma_2^2(t) \]

\[ \tilde{B}(t) = \frac{\sigma_1(t)}{\tilde{\sigma}(t)} B_1(t) - \frac{\sigma_2(t)}{\tilde{\sigma}(t)} B_2(t). \]
3.2 Impermanent Loss

Impermanent loss is the value of a position in staked tokens minus the value of an unstaked position that initially had the equivalent value as the staked position, and then the difference divided by the value of the unstaked position. At time \( t \), let \( V_{\text{staked}}(t, s) \) denote the dollar value staked in the pool at time \( s \geq t \), and let \( V_{\text{held}}(t, s) \) denote the value of an un-staked position that has an equal dollar amount as time \( t \); that is \( V_{\text{staked}}(t, t) = V_{\text{held}}(t, t) \). For a time increment \( \Delta t > 0 \), at time \( t + \Delta t \) the impermanent loss is defined as follows.

**Definition 3.1.** For a time increment \( \Delta t = 1/n \) where \( n \) is a positive integer, the impermanent loss from time \( t \) to time \( t + \Delta t \) is the staked value minus the held value, divided by the held value, as follows:

\[
\Delta IL_n(t) := \frac{V_{\text{staked}}(t, t + \Delta t) - V_{\text{held}}(t, t + \Delta t)}{V_{\text{held}}(t, t + \Delta t)}. \tag{3.8}
\]

In terms of \( \Delta IL_n(t) \), which is defined by equation (3.8), the total impermanent loss up to time \( t \) is the summation to time \( t \),

\[
IL_n(t) := \sum_{i=0}^{\lfloor \frac{t}{\Delta t} \rfloor - 1} \Delta IL_n(t_i),
\]

where \( t_i = i\Delta t = i/n \). Total impermanent loss in continuous time is obtained by taking the limit as \( n \) goes to infinity,

\[
IL(t) := \lim_{n \to \infty} IL_n(t). \tag{3.9}
\]

In discrete time it is straightforward to show that \( \Delta IL_n(t) \leq 0 \), see Heimbach et al. (2022), and in continuous time it is also true that impermanent loss is always nonpositive.

**Proposition 3.1.** In continuous time, impermanent loss is the differential of total impermanent loss \( IL(t) \), which is defined in equation (3.9), and which is equal to \(-\frac{1}{8}\) times the variance of the relative price times the length of time increment,

\[
dIL(t) = -\frac{\hat{\sigma}^2(t)}{8} dt, \tag{3.10}
\]

where \( \hat{\sigma}(t) = \sqrt{\sigma_1^2(t) - 2\rho\sigma_1(t)\sigma_2(t) + \sigma_2^2(t)} \) is the volatility of the relative price \( R(t) \) seen in equations (3.6) and (3.7). From equation (3.10) it is clear that \( dIL(t) \leq 0 \) for all \( t \geq 0 \).

**Proof.** See Appendix A.1. \( \square \)
4 Model-Free Valuation

Within the stochastic volatility framework given in Section 3, the total impermanent loss is negative 1/8 times the realized variance of the relative price. From here forward, let $\mathbb{E}^Q$ denote risk-neutral expectation with USD as the numéraire. The risk-neutral valuation of impermanent loss is equivalent to the valuation of a variance swap. For time window $[0, T]$ the expected realized variance is equal to twice the total returns minus the log contract. By applying Ito’s lemma on $\ln R(t)$ and taking integrals on both sides, together with equation (3.10), one obtains for the time window $[0, T]$ the expected realized variance, which is equal to twice the total returns minus the log contract

$$
\mathbb{E}^Q \left[ \int_0^T \hat{\sigma}^2 (t) \, dt \right] = 2 \mathbb{E}^Q \left[ \int_0^T \frac{dR(t)}{R(t)} - \log \left( \frac{R(T)}{R_F} \right) \right],
$$

where $R_F = \mathbb{E}^Q [R(T)]$. Ideally, the right-hand side of (4.1) can be computed with a model-free method, giving a purely market-driven prediction of variance. In many pricing problems, the expected total returns are equivalent to an interest-rate swap and can be valued with a short-term zero-coupon bond yield. For the log contract, the formula of Carr and Madan (1999) gives the valuation in terms of a portfolio of European options on $R(T)$; the same valuation can be obtained using $R(T)$’s risk-neutral density given by the formula of Breeden and Litzenberger (1978). However, there are two main difficulties when performing this model-free valuation for token pairs. The first is that there is no market for options on the tokens’ relative price. The second is that the relative price is not a risk-neutral martingale, which means that $R(t)$’s expected rate of return cannot be equated to the short-term interest rate. To manage the first difficulty we compute a joint density from existing options on the separate tokens. To manage the second we will employ a change of numéraire.

4.1 Marginal Densities

Centralized exchanges offer European options on the individual token. For example, Deribit offers options on BTC and ETH, each of which is cash settled in terms of their respective underlying, and with margins also settled in the respective underlying. Using the density formula of Breeden and Litzenberger (1978), options with a fixed maturity and a continuum of strikes provide the risk-neutral distribution on the future states of the underlying token. For a token pair, the density formula provides the two marginal densities and then a further step must be taken to estimate their dependency structure for their joint distribution. This joint distribution is exactly what we would need to value the log contract on $R(T)$. 

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Let $C_1(T, K)$ and $C_2(T, K)$ denote European call option prices on $P_{1}(T)$ and $P_{2}(T)$, respectively. The marginal densities are
\[ f_{i}(T, K) = e^{rT} \frac{\partial^{2}}{\partial K^{2}} C_{i}(T, K) \quad \text{for} \quad i = 1, 2. \]

In practice, the discreteness of strike prices requires us to interpolate points in the interior of the strikes’ range and also to extrapolate the density into the tail. We do this interpolation at the level of the implied volatility,
\[ f_{i}(T, K; \theta) = e^{rT} \frac{\partial^{2}}{\partial K^{2}} C^{BS}(T, K, \dot{\sigma}_{i}(T, K; \theta)) \quad \text{for} \quad i = 1, 2, \]
where $\dot{\sigma}_{i}(T, K; \theta)$ is a function with parameter $\theta$ that has been fitted to the implied volatility smile of quoted options, and where $C^{BS}(T, K, \sigma)$ is the Black-Scholes call price with volatility-input $\sigma$. It is shown in Figlewski (2018) that improved estimation of the risk-neutral density is obtained by parametrically fitting the implied volatility smile and then differentiating the option price. In particular, a polynomial of degree 4 or 5 is usually sufficient for fitting quoted implied volatilities, and some type of heavy-tailed parametric distribution is used to extrapolate into the tail. Figure 4.1 shows the 4th-order polynomial fitted to the quoted BTC implied volatilities from Deribit with $T = 30$ days on May, 18 of 2023 at the beginning of the day (UTC); the tails are extrapolated with a log-normal density with volatility parameter taken to be the very last quoted implied volatility. The 4th-order polynomial fit is convenient because the implied density follows from a simple application of the chain rule when computing the second $K$ partial derivative of $C^{BS}(T, K, \dot{\sigma}_{i}(T, K; \theta))$ for $i = 1, 2$.

4.2 Optimal Joint Density

The densities $f_{1}(T, K)$ and $f_{2}(T, K)$ can be combined in a computation to obtain a joint risk-neutral distribution for $(P_{1}(T), P_{2}(T))$. We assume that we have the joint physical density, and then the estimation of the joint pricing kernel uses an optimization objective that adheres to the principles of arbitrage theory. In particular, any acceptable estimation should respect the law-of-one-price by preserving the observed market prices (i.e., any solution should have marginals equal to $f_{1}$ and $f_{2}$), and also should not introduce any inter-market arbitrage opportunities. A financially meaningful objective is to minimize the Hansen and Jagannathan (1991) (HJ) upper bound. The HJ bound says that no portfolio can have a Sharpe ratio in excess of the standard deviation of any pricing kernel. This is
Figure 4.1: BTC Implied Volatility and the Estimated Risk-Neutral Density. The plot shows the BTC implied volatility surface. The quoted volatilities are fit with a 4th-degree polynomial, and extrapolation of the tail densities uses a log-normal density.

expressed as

\[
\sup_{\Pi: \text{std}(\Pi) > 0} \frac{\text{EII}}{\text{std}(\Pi)} \leq \inf_{M \in \mathcal{M}(\mu, \nu)} \text{std}(M),
\]

(4.2)

where \(\Pi\) denotes the excess return on a portfolio, \(\mathcal{M}(\mu, \nu)\) denotes the family of pricing kernels that respect marginals \(\mu\) and \(\nu\), and \(\text{std}(\cdot)\) denotes standard deviation. The left-hand side of equation (4.2) is a portfolio optimization, which we refer to as our primal problem. The right-hand is an optimal pricing problem, which we refer to as our dual problem. If there are \(\Pi\) and \(M\) such that \(\frac{\text{EII}}{\text{std}(\Pi)} = \text{std}(M)\), then there is no duality gap. Equation (4.2) is derived from the Cauchy-Schwartz inequality for the covariance between \(\Pi\) and \(M\), \(\text{cov}(\Pi, M) \geq -\text{std}(\Pi)\text{std}(M)\). The Cauchy-Schwartz is an equality if and only if \(\Pi\) is a scalar (deterministic) multiple of \(M\). For complete markets, \(\mathcal{M}(\mu, \nu)\) is a singleton set with a unique \(M\) that is replicated by a portfolio of Arrow-Debreu securities, and hence there is no duality gap. For incomplete markets, there is no duality gap if the optimal \(M\) is in the span of investible portfolios, which is in general not the case, but when the individual assets each have a complete set of options then there is no gap because these options form a portfolio that replicates the pricing kernel.

Let \(q(x, y)\) be a density such that pricing kernel \(M(x, y) = \frac{q(x, y)}{p(x, y)}\). The optimization for computing
the HJ upper bound is a quadratic program.

\[
\begin{align*}
\text{minimize} & \quad \int \int \left( \frac{q(x, y)}{p(x, y)} \right)^2 p(x, y) \, dx \, dy, \\
\text{subject to} & \quad \int q(x, y) \, dy = f_1(T, x), \\
& \quad \int q(x, y) \, dx = f_2(T, y).
\end{align*}
\] (4.3)

where \( p(x, y) \) is the physical density, and \( q \sim p \) denotes the set of densities such that \( q(x, y) > 0 \) if and only if \( p(x, y) > 0 \) for all \( (x, y) \). The existence and uniqueness of solutions, and the existence of a Sharpe-optimal portfolio, are proven in Guasoni and Mayerhofer (2020). Trivially, there always exists a solution to (4.3), namely, the product measure \( f_1(T, x)f_2(T, y) \) under which the two tokens are independent. In general, the problem is convex and an approach to solving is to use a Lagrangian,

\[
L = \int \int q^2(x, y) \frac{1}{p(x, y)} \, dx \, dy - \int \lambda_1(x) \left( \int q(x, y) \, dy - f_1(T, x) \right) \, dx - \int \lambda_2(y) \left( \int q(x, y) \, dx - f_2(T, y) \right) \, dy
\]

from which the unique non-negative solution is expressed as \( q(x, y) = \frac{1}{2}(\lambda_1(x) + \lambda_2(y))p(x, y) \), where \( \lambda_1(x) \) and \( \lambda_2(y) \) are Lagrange multiplier functions, and where the optimal pricing kernel is

\[ M(x, y) = \frac{1}{2} \left( \lambda_1(x) + \lambda_2(y) \right). \]

Assuming that every point \( (x, y) \) has a positive physical probability of occurring, then this pricing kernel is arbitrage-free if and only if it is strictly positive.

**Proposition 4.1.** Assume \( p(x, y) > 0 \) for all \( (x, y) \). The optimal pricing kernel from the HJ upper bound in (4.3) is arbitrage-free if and only if \( \lambda_1(x) + \lambda_2(y) > 0 \) for all \( (x, y) \).

We do not show proof for Proposition 4.1 because it is a direct invocation of the fundamental theorem of asset pricing. Regarding the Lagrangian, it has been written without a non-negativity constraint because avoidance of arbitrage requires a solution to be in the interior, and therefore this additional constraint would be inactive.

There is also replicability of the Lagrange multiplier functions:

**Proposition 4.2.** Given optimal Lagrange multipliers \( \lambda_1(x) \) and \( \lambda_2(y) \) from minimization of the HJ upper bound in (4.3), \( \lambda_1(x) \) can be replicated by a portfolio European call and put options on \( P_1(T) \), forward contracts on \( P_1(T) \), and cash. Similarly, \( \lambda_2(y) \) can be replicated by a portfolio of European call and put options on \( P_2(T) \), forward contracts on \( P_2(T) \), and cash.
Proof. Regularity of the Lagrange multiplier functions as proved in Guasoni and Mayerhofer (2020), where if \( \mu \) and \( \nu \) are sufficiently differentiable then so is the solution to the dual problem. Given this regularity, the formula of Carr and Madan (1999) is used to replicate the Lagrange multipliers,

\[
\lambda_1(x) = \lambda_1(x_0) + \lambda'_1(x_0)(x-x_0) + \int_{x_0}^{\infty} \lambda''_1(K)(K-x)^+ dK + \int_{x_0}^{\infty} \alpha''_1(K)(x-K)^+ dK
\]

\[
\lambda_2(y) = \lambda_2(y_0) + \lambda'_2(y_0)(y-y_0) + \int_{y_0}^{\infty} \lambda''_2(K)(K-y)^+ dK + \int_{y_0}^{\infty} \lambda''_2(K)(y-K)^+ dK
\]

where \( x_0 \) and \( y_0 \) are chosen reference points, usually the forward prices of \( P_1(T) \) and \( P_2(T) \), respectively. For \( x, \lambda_1(x_0) \) is a position in cash, \( \lambda'_1(x_0)(x-x_0) \) is the net position of a position in \( \lambda_1(x_0) \)-many forward contracts, and the integrals are portfolios of out-of-the-money calls and puts; the same breakdown applies for \( y \) and \( \lambda_2(y) \).

Using Proposition 4.2 we can construct a portfolio that is a (negative) scalar multiple of the pricing kernel and therefore has a Sharpe ratio equal to the the right-hand side of (4.3). That is, with cash, forward contracts, and calls and puts, we construct a portfolio \( \Pi \) whose excess returns are

\[
\Pi(x, y) = -\frac{1}{2}(\lambda_1(x) + \lambda_2(x)) = -M(x, y).
\]

For this \( \Pi \) we have \( \frac{\mathbb{E}||\Pi||}{\text{std}(\Pi)} = \text{std}(M) \), showing that there is no duality gap in (4.2). For implementation details, Appendix B shows a sparse quadratic program when mass functions \( f_1(T, K)\Delta K \) and \( f_2(T, K)\Delta K \) are the inputs.

After optimizing the HJ bound, the density we obtain could be used to compute prices for European options on \( R(T) = P_1(T)/P_2(T) \). One usage of these prices is to insert them into the formula of Carr and Madan (1999) to obtain the valuation of the log contract. However, this log-contract valuation will not give us a risk-neutral expectation of realized variance, as we will explain in the next section.

4.3 Change of Numéraire

The relative price \( R(t) \) is a marginal rate of substitution for a liquidity pool, and so unless the base token is pegged to the local currency it is not accurate to say that \( R(t) \) grows at the risk-free rate. Specifically, it is difficult to calculate \( \mathbb{E}^Q \int_0^T \frac{dR(t)}{R(t)} \) in equation (4.1). A change of numéraire resolves this issue and simplifies the computation for the swap valuation. This change of numéraire should be carried out in the base token, which in our case is \( P^2(t) \). For example, we could make a change of
numéraire to the Ethereum price. The following change of measure is introduced,

\[
\left. \frac{d\tilde{Q}}{dQ} \right|_{T} = \frac{P_2(T)}{E^Q P_2(T)} .
\]

By Girsanov’s Theorem, under \( \tilde{Q} \), \( \tilde{B}_2(t) := B_2(t) - \int_0^t \sigma_2(s)ds \) and \( \tilde{B}_1(t) := B_2(t) - \rho \int_0^t \sigma_2(s)ds \) are standard Brownian motions, and

\[
dR(t) = R(t) \left( \sigma_1(t) d\tilde{B}_1(t) - \sigma_2(t) d\tilde{B}_2(t) \right),
\]

that is, \( R(t) \) is a \( \tilde{Q} \)-martingale on time interval \([0, T]\). Details on this Girsanov change of measure are given in Appendix A.2. Moreover, the valuation of the variance swap rate under this change of measure is only the expectation of the log contract

\[
E^Q \left( \sigma \int_0^T \tilde{\sigma}^2(t)dt \right) \quad \text{and} \quad E^Q \log \left( \frac{R(T)}{R_0} \right),
\]

which can be easily calculated with the joint risk-neutral density that was obtained the HJ optimization in Section 4.2. For the variance swap valuation under \( \tilde{Q} \), the elemental pieces for the formula of Carr and Madan (1999) or the formula of Breeden and Litzenberger (1978), are the following European call and put options,

\[
\begin{align*}
\tilde{E}^Q(R(T) - K)^+ &= e^{-rT} P_2(0) \int \int y \left( \frac{x}{y} - K \right)^+ q(x, y) dx dy, \\
\tilde{E}^Q(K - R(T))^+ &= e^{-rT} P_2(0) \int \int y \left( K - \frac{x}{y} \right)^+ q(x, y) dx dy,
\end{align*}
\]

from which we can write

\[
\tilde{E}^Q \int_0^T \tilde{\sigma}^2(t)dt = 2 \left( \int_{R(0)}^{R(T)} \frac{\tilde{E}^Q(K - R(T))^+}{K^2} dK + \int_{R(T)}^{R(T)} \frac{\tilde{E}^Q(R(T) - K)^+}{K^2} dK \right).
\]

### 4.4 Margrabe Formula and Implied Correlation

The variance swap valuation obtained under \( \tilde{Q} \) using the options in (4.4) is useful, but it is just a single number that is perhaps not the best gauge for the level risk for exposure to \( R(T) \). Specifically, the expected variance \( \tilde{E}^Q \int_0^T \tilde{\sigma}^2(t)dt \) is a rate that could be dominated by high implied volatilities in the individual tokens, which is certainly informative, but a more precise understanding of the joint-token risk is found if analyze implied correlations. To compute the implied correlation we will use spread options and the Margrabe formula.

The option valuations in (4.4) are proportional to spread option value under the original risk-neutral measure. Let us denote call and put spread options as

\[
\begin{align*}
C_{\text{spread}}(T, K) &= e^{-rT} \tilde{E}^Q(P_1(T) - KP_2(T))^+ \\
P_{\text{spread}}(T, K) &= e^{-rT} \tilde{E}^Q(KP_2(T) - P_1(T))^+,
\end{align*}
\]

Electronic copy available at: https://ssrn.com/abstract=4811111
and notice that these are equal to the \( \tilde{Q} \) options in (4.4) times the numéraire token, \( C_{\text{spread}}(T, K) = P_2(0)\tilde{E}^Q(R(T) - K)^+ \) and \( P_{\text{spread}}(T, K) = P_2(0)\tilde{E}^Q(K - R(T))^+ \). In a constant-volatility model, the Margrabe formula explicitly gives the price for a spread option Margrabe (1978). Using the Margrabe formula, an implied volatility is \( \tilde{s}(T, K) \) such that

\[
C_{\text{spread}}(T, K) = P_1(0)\Phi(d_1) - K P_2(0)\Phi(d_2)
\]

where

\[
d_1 = \frac{\log(P_1(0)/P_2(0)) + \frac{1}{2}\tilde{s}^2(T, K)T}{\tilde{s}(T, K)\sqrt{T}}
\]

\[
d_2 = d_1 - \tilde{s}(T, K)\sqrt{T}.
\]

For the put spread we have \( P_{\text{spread}}(T, K) = KP_2(0)\Phi(-d_2) - P_1(0)\Phi(-d_1) \). From \( \tilde{s}(T, K) \) we compute a dispersion index,

\[
\hat{\rho}(t, K) = \frac{\hat{\sigma}_1^2(T) + \hat{\sigma}_2^2(T) - \tilde{s}^2(T, K)}{2\hat{\sigma}_1(T)\hat{\sigma}_2(T)}
\]

where \( \hat{\sigma}_1^2(T) \) and \( \hat{\sigma}_2^2(T) \) are implied volatility indices computed from the options on the \( P_1(T) \) and \( P_2(T) \), respectively.

5 Data Description

This section provides a general description of data sources and construction of variables in Section 5.1 for liquidity pools and in Section 5.2 for the option-implied variables.

5.1 Liquidity Pool Data

We analyze daily data from Uniswap V3, which is the third version of the Uniswap decentralized exchange protocol. We source data from the 250 Uniswap pools with the largest TVLs, using the methodology provided by The Graph. The Graph is a protocol that helps in accessing information on the Ethereum blockchain by allowing users to use a query language called GraphQL. We then calculate the impermanent loss and historical Annual Percentage Rates (APRs) for each pool for the period of 5-2021 to 11-2023.

For the calculation of the impermanent loss over the last \( n \) days we adapt the equation given in Heimbach et al. (2022)

\[
IL(t, n) := \frac{V_{\text{Pool}}(t) - V_{\text{Hold}}(t)}{V_{\text{Hold}}(t)} = \frac{2\sqrt{\frac{S_t}{S_{t-n}}} - 1}{1 + \frac{S_t}{S_{t-n}} - 1}
\]
The estimated returns for liquidity provision are quoted in terms of APRs, which denote the annualized fraction of fees collected from the liquidity takers \((fees_t)\) over some period divided by the total volume of the pool, with the latter being measured in \(total\ value\ locked\ (TVL)\). The APR is calculated as

\[
APR_t = \frac{fees_t}{TVL_t}.
\]

Depending on the use case we annualized the respective measures to make them comparable to their implied quantities explained next.

## 5.2 Option Data and Risk-Neutral Moments

We obtain option data through amberdata\(^4\), encompassing hourly surface data for standardized maturities and moneyness levels. The data is taken from Deribit, which is the largest exchange for trading cryptocurrency options.\(^5\) On Deribit options are cash-settled and quoted directly in the respective cryptocurrency and not in USD. We consider BTC and ETH as underlying assets. The surface data (for each standardized maturity) consists of 5 moneyness levels (measured by delta) per call or put. In the next step, we interpolate across moneyness to obtain a smooth surface which allows us to extract the risk-neutral distribution for each underlying following Breeden and Litzenberger (1978)

\[
\frac{\partial^2 C(T, K)}{\partial K^2} \bigg|_{K=p} = e^{-rT} f_i(T, p).
\]

### 5.2.1 Implied Impermanent Loss

As shown in Section 4.4, we can construct a new measure as \(\frac{\partial^2}{\partial x^2} \bigg|_T = \frac{\partial^2 C(T, K)}{\partial K^2} \bigg|_{K=p} e^{-rT} f_i(T, p)\) to obtain a risk-neutral valuation of impermanent loss,

\[
\tilde{E}IL(T) = \frac{1}{8T} \int_0^T \sigma^2(t) dt = -\frac{1}{4T} \tilde{E}\log(R(T)/R(0)) = -\frac{1}{4T} \int \log \left(\frac{x}{(R(0)y)}\right) q_T(x, y) dx dy
\]

where \(q_T(x, y)\) is the optimal joint density as described in Section 4.2, computed with options of maturity \(T\). For a discrete mesh \(\{(x_i, y_j)_{i,j}\}\) with increments \(\Delta x\) and \(\Delta y\) between mesh points, upon

\(^4\)https://www.amberdata.io/

\(^5\)The option trading volume across different exchanges is displayed in Figure C.1.
which we have a mass function \( q_{i,j} = \frac{y_{i} \times q_{i,j}}{F_{y}} \) where \( q_{i,j} \approx q_{T}(x_{i}, y_{j}) \Delta x \Delta y \) and \( F_{y} = \sum_{i,j} y_{j} q_{i,j} \), there is the following estimate of impermanent loss,

\[
\tilde{\text{EIL}}(T) \approx -\frac{1}{4T} \sum_{i} \sum_{j} q_{i,j} \log \left( \frac{x_{i}/(y_{j} R(0))}{\cdot} \right).
\]

### 5.2.2 Implied Variances

We calculate the risk-neutral IVs for each underlying for a maturity of 30 days following Martin (2016) (for the gross return) as

\[
SVIX^2_{i} = \frac{2}{(T - t)S_{t}^{2}} \int_{0}^{F_{t,T}} \text{put}_{t,T}(K) dK + \int_{F_{t,T}}^{\infty} \text{call}_{t,T}(K) dK.
\]

In the computations, we discretize the integral via the trapezoidal rule by applying our interpolated volatility surface.

### 5.2.3 Implied Correlation

The implied correlation (IC) (for a maturity of 30 days) is calculated as follows

\[
IC_{BTC,ETH} = \frac{IV^2_{BTC} + IV^2_{ETH} - IV^2(K, T)}{2IV_{BTC}IV_{ETH}}
\]

where \( IV^2(K, T) \) is the implied volatility obtained by inverting the Margrabe (1978) formula for the spread option where \( K \) is set such that \( KP^2/P^1 = 1.1 \) (for BTC-ETH: \( K \approx 37 - \text{OTM Call} \)).

### 6 Empirical Analysis

Within this section, we conduct an empirical analysis aimed at unveiling and presenting the previously discussed metrics outlined in Section 5. The analysis covers a broad cross-section of lipidity pools for various token pairs for any Ethereum-based tokens on Uniswap. We analyze implied impermanent loss and its components for the BTC-ETH token pair using options data, finding that higher implied token variance increases impermanent loss, while higher implied correlation decreases it due to the reduced impact of positive token comovement on relative prices. In the next step, we compare our implied variables with realized counterparts from historical data. Finally, we investigate liquidity provision returns, emphasizing impermanent loss as a critical risk factor shaping the risk-return relationship. As LPs deposit assets into pools, token price changes trigger impermanent loss, offset by increased trading activity, resulting in higher APR and rewarding LP participation. The
Fama-Macbeth (1973) cross-sectional regressions confirm the significance of the implied impermanent loss and its drivers. Implied impermanent loss explains the cross-section with consistent results in a multivariate setting. LPs favor high positive correlations among tokens to mitigate impermanent loss, resulting in a negative price of risk for implied correlation.

6.1 Uniswap - Risks and Rewards

The impermanent loss on Uniswap represented as a time-series mean for individual pools, is visually presented in Figure 6.1 as a histogram. The annualized impermanent loss across the pools is on average at around -10.3%, with extreme values up to −70%. Therefore, the impermanent loss should be treated as a nonnegligible risk.

![Histogram of the Impermanent Loss – Uniswap Pools](https://ssrn.com/abstract=4811111)

**Figure 6.1: Histogram: Impermanent Loss – Uniswap Pools.** The figure shows the histogram of the (average) impermanent loss (annualized) of the Uniswap pools. The dotted line represents the average impermanent loss. The data is sampled daily, and the sample period ranges from 05-2021 to 11-2023. The data is winzorized at the 5% quantile.

Following that, we present the average APRs for the Uniswap pools. Figure 6.2 showcases the histogram of the time-series average gross APRs (Panel (a)) and net APRs (calculated as gross APR minus impermanent loss) (Panel (b)). The mean gross APR is approximately 15%, with outliers reaching nearly 50%. However, due to substantial impermanent loss, the average net APR decreases to only 5%, and there are instances of notably negative net APRs, extending to as low as −30%.

In Figure 6.3, the gross and net APRs for the Uniswap pool (Uniswap), calculated as the cross-sectional average across 250 individual pools, are depicted over time. The graph illustrates the volatility of both gross and net APRs. Notably, the gross APR has exhibited a consistent decline, whereas the net APR has remained relatively stable, hovering around 5% since June 2022.
Figure 6.2: Histogram: Uniswap Pool APRs. The figure shows the histogram of the (average) APRs of the Uniswap pools. In Panel (a) the gross APRs are displayed, and in Panel (b), the net APRs (gross APR – impermanent loss). The dotted line represents the average. The data is sampled daily, and the sample period is from 05-2021 to 11-2023. The data is winzorized at the 5% quantile.

Figure 6.3: Uniswap Pool APRs. The figure shows the cross-sectional average of the APRs of the Uniswap pools (Uniswap). In Panel (a), the gross APRs are displayed, and in Panel (b) the net APRs (gross APR – impermanent loss). The dotted line represents the average. The data is sampled daily, and the sample period is from 05-2021 to 11-2023. The data is winzorized at the 1% quantile. In the plots, the five-day moving average is depicted.

6.2 The Implied Impermanent Loss and its Drivers

Figure 6.4 displays the option implied variables together with their realized counterparts. From Panel (a) it is visible that the implied impermanent loss comoves with the realized counterpart which is an extremely noisy time series if not smoothed.\(^6\) \(IV\) and \(RV\) for BTC (Panel (b)) and ETH (Panel (c)). Panel (d) depicts the dynamics of implied correlation over time for a 30-day maturity together with the realized correlation. Implied correlation, being a bounded variable, displays (relatively) smaller fluctuations than the implied volatilities.

Table 6.1 reports summary statistics for the implied variables. The implied impermanent loss is about 0.14. Moving on to implied volatility, the second and third columns represent \(IV_{BTC}\) and \(IV_{ETH}\)

\(^6\)The realized impermanent loss smoothed with only 7 or 2 days is displayed in Figure C.2. As visible the realized impermanent loss can be substantially larger than the implied impermanent loss.
Figure 6.4: $IV – BTC$ and $ETH$. The figure shows the implied impermanent loss ($IL^Q$) and realized impermanent loss ($IL^P$) for BTC and ETH (Panel (a)), the implied volatility ($IV$) and realized volatility ($RV$) for BTC (Panel (b)) and ETH (Panel (c)), and the implied correlation ($IC$) and realized correlation (Panel (d)) ($RC$) for BTC and ETH. The implied IL is calculated following equation (5.2.1) and for a maturity of 30 days. The daily realized impermanent loss is calculated using equation (5.1), smoothed with a 30-day moving average, annualized, and expressed in terms of volatility. The $IV$ is calculated following Martin (2016), equation (5.2.2), and for a maturity of 30 days. $RV$ is calculated over a backward-looking window of 30 days. The $IC$ is calculated following equation (5.2.3) and for a maturity of 30 days. $RC$ is calculated over a backward-looking window of 30 days. The data is sampled daily, and the sample period is from 05-2021 to 11-2023.

The average value of 0.885 for the implied correlation ($IC_{BTC,ETH}$), suggests a strong average correlation between the price movements of BTC and ETH as perceived by the derivatives market. For the realized variables the monthly annualized impermanent loss is approximately 0.086 (compared to 0.14 under $Q$). In terms of realized volatility, the mean values for $IV_{BTC}$ and $IV_{ETH}$ in the second and third columns are 0.59 and 0.75, compared to 0.68 and 0.81, respectively. Notably, the realized correlation ($RC_{BTC,ETH}$) exhibits an average realized correlation of 0.85 (as compared to 0.88 for implied correlation) between the price movements of BTC and ETH,
as measured from realized data. In summary, there are differences in the mean values derived from options data or from realized data for the impermanent loss, volatilities, and correlation.

<table>
<thead>
<tr>
<th></th>
<th>$IL^Q_{BTC,ETH}$</th>
<th>$IV_{BTC}$</th>
<th>$IV_{ETH}$</th>
<th>$IC_{BTC,ETH}$</th>
<th>$IL^P_{BTC,ETH}$</th>
<th>$RV_{BTC}$</th>
<th>$RV_{ETH}$</th>
<th>$RC_{BTC,ETH}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.140</td>
<td>0.685</td>
<td>0.815</td>
<td>0.885</td>
<td>0.086</td>
<td>0.593</td>
<td>0.750</td>
<td>0.859</td>
</tr>
<tr>
<td>Std</td>
<td>0.065</td>
<td>0.189</td>
<td>0.278</td>
<td>0.068</td>
<td>0.072</td>
<td>0.214</td>
<td>0.334</td>
<td>0.078</td>
</tr>
<tr>
<td>Per 10</td>
<td>0.066</td>
<td>0.427</td>
<td>0.410</td>
<td>0.800</td>
<td>0.010</td>
<td>0.339</td>
<td>0.367</td>
<td>0.741</td>
</tr>
<tr>
<td>Median</td>
<td>0.132</td>
<td>0.684</td>
<td>0.816</td>
<td>0.898</td>
<td>0.063</td>
<td>0.586</td>
<td>0.725</td>
<td>0.874</td>
</tr>
<tr>
<td>Per 90</td>
<td>0.218</td>
<td>0.920</td>
<td>1.118</td>
<td>0.952</td>
<td>0.230</td>
<td>0.848</td>
<td>1.095</td>
<td>0.938</td>
</tr>
</tbody>
</table>

Table 6.1: Summary Statistics – Implied and Realized Variables. The table reports the summary statistics (time-series mean, median, standard deviation, and the 10% and 90% percentiles) for the implied impermanent loss ($IL^Q_{BTC,ETH}$) (equation (5.2.1)), the implied volatilities of BTC ($IV_{BTC}$) and ETH ($IV_{ETH}$) (equation (5.2.2)), the implied correlation ($IC_{BTC,ETH}$) (equation (5.2.3)), the realized impermanent loss ($IL^P_{BTC,ETH}$) (equation (5.1)) annualized and in volatility terms, the realized volatilities of BTC ($RV_{BTC}$) and ETH ($RV_{ETH}$), and the realized correlation ($RC_{BTC,ETH}$). The implied variables are constructed for a maturity of 30 calendar days while the realized variables are constructed over a backward-looking window of 30 calendar days. The sample period is from 05-2021 to 11-2023.

<table>
<thead>
<tr>
<th></th>
<th>$IL^Q_{BTC,ETH}$</th>
<th>$IV_{BTC}$</th>
<th>$IV_{ETH}$</th>
<th>$IC_{BTC,ETH}$</th>
<th>$IL^P_{BTC,ETH}$</th>
<th>$RV_{BTC}$</th>
<th>$RV_{ETH}$</th>
<th>$RC_{BTC,ETH}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$IL^Q_{BTC,ETH}$</td>
<td>1.000</td>
<td>0.827</td>
<td>0.881</td>
<td>-0.605</td>
<td>0.363</td>
<td>0.825</td>
<td>0.854</td>
<td>-0.345</td>
</tr>
<tr>
<td>$IV_{BTC}$</td>
<td>0.827</td>
<td>1.000</td>
<td>0.948</td>
<td>-0.272</td>
<td>0.394</td>
<td>0.807</td>
<td>0.799</td>
<td>-0.153</td>
</tr>
<tr>
<td>$IV_{ETH}$</td>
<td>0.881</td>
<td>0.948</td>
<td>1.000</td>
<td>-0.240</td>
<td>0.400</td>
<td>0.783</td>
<td>0.829</td>
<td>-0.121</td>
</tr>
<tr>
<td>$IC_{BTC,ETH}$</td>
<td>-0.605</td>
<td>-0.272</td>
<td>-0.240</td>
<td>1.000</td>
<td>-0.117</td>
<td>-0.441</td>
<td>-0.392</td>
<td>0.558</td>
</tr>
<tr>
<td>$IL^P_{BTC,ETH}$</td>
<td>0.363</td>
<td>0.394</td>
<td>0.400</td>
<td>-0.117</td>
<td>1.000</td>
<td>0.338</td>
<td>0.353</td>
<td>-0.073</td>
</tr>
<tr>
<td>$RV_{BTC}$</td>
<td>0.825</td>
<td>0.807</td>
<td>0.783</td>
<td>-0.441</td>
<td>0.338</td>
<td>1.000</td>
<td>0.969</td>
<td>0.003</td>
</tr>
<tr>
<td>$RV_{ETH}$</td>
<td>0.854</td>
<td>0.799</td>
<td>0.829</td>
<td>-0.392</td>
<td>0.353</td>
<td>0.960</td>
<td>1.000</td>
<td>0.037</td>
</tr>
<tr>
<td>$RC_{BTC,ETH}$</td>
<td>-0.345</td>
<td>-0.153</td>
<td>-0.121</td>
<td>0.558</td>
<td>-0.073</td>
<td>0.003</td>
<td>0.037</td>
<td>1.000</td>
</tr>
</tbody>
</table>

Table 6.2: Contemporaneous Correlations. The table reports the correlation coefficient for the implied impermanent loss ($IL^Q_{BTC,ETH}$) (equation (5.2.1)), the implied volatilities of BTC ($IV_{BTC}$) and ETH ($IV_{ETH}$) (equation (5.2.2)), the implied correlation ($IC_{BTC,ETH}$) (equation (5.2.3)), the realized impermanent loss ($IL^P_{BTC,ETH}$) (equation (5.1)) annualized and in volatility terms, the realized volatilities of BTC ($RV_{BTC}$) and ETH ($RV_{ETH}$), and the realized correlation ($RC_{BTC,ETH}$). The implied measures are constructed for a maturity of 30 calendar days while the realized measures are constructed over a backward-looking window of 30 calendar days. The sample period is from 05-2021 to 11-2023.

Table 6.2 presents the time-series correlation of the various variables in levels. The correlation between the $IL^Q$ and its realized counterpart ($IL^P$) is about 0.36. Naturally, there exists a strong comovement between the implied and the realized volatility (the correlation is around 0.80 for BTC and ETH). The time-series correlation between $IC$ and $RC$ is 0.70 (0.23) in levels (changes). Interestingly, the time-series correlations between the implied correlation and the implied volatilities are negative (or low); for example, the correlation with the implied volatility for BTC is −0.27 in levels (or 0.11 in differences) and similar for ETH in levels −0.24 (or 0.16 in differences); the correlation with the implied permanent loss is −0.60 in levels (−0.53 in differences). The correlation between the implied volatility and the implied permanent loss is 0.82 (0.88) for BTC (ETH) in levels while for the changes the time-series correlation is only 0.09 (0.24) for BTC (ETH). The pronounced interplay of positive and negative relationships among distinct components, with the implied permanent loss, highlights the critical importance of factoring in the univariate and multivariate distribution of individual tokens.
during the estimation process.

In our empirical evaluation of the impact of various components on implied impermanent loss, we employ straightforward regressions using daily levels of implied impermanent loss and corresponding levels in its components. To underscore the individual contributions of each component, we initially isolate the most crucial variable, followed by the most influential pair of variables. Subsequently, we include all three explanatory variables in a hierarchical regression approach. The outcomes of these analyses are presented in Table 6.3: The majority of implied impermanent loss is accounted for by the implied volatility of ETH, exhibiting $R^2$ values of nearly 70%. Following closely, the implied correlation contributes approximately 20% to the $R^2$. In contrast, the implied volatility of BTC plays a lesser role, explaining only an additional 1% to 2% of the implied impermanent loss. In accordance with our stylized framework, the implied volatility of ETH and the implied correlation exert opposing influences on the implied impermanent loss.\(^7\)

<table>
<thead>
<tr>
<th></th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>-0.040</td>
<td>0.297</td>
<td>0.304</td>
</tr>
<tr>
<td></td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>$IV_{BTC}$</td>
<td>-</td>
<td>-</td>
<td>-0.042</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>-</td>
<td>0.000</td>
</tr>
<tr>
<td>$IV_{ETH}$</td>
<td>0.233</td>
<td>0.189</td>
<td>0.219</td>
</tr>
<tr>
<td></td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>$IC_{BTC,ETH}$</td>
<td>-</td>
<td>-0.352</td>
<td>-0.353</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>$R^2$</td>
<td>69.880</td>
<td>91.918</td>
<td>92.129</td>
</tr>
</tbody>
</table>

Table 6.3: Drivers of the Implied Impermanent Loss. The table reports the results (slope, p-value, and the $R^2$s) of the regressions of daily implied impermanent loss (equation (5.2.1)) on its components, namely, the $IV_{BTC}$, $IV_{ETH}$ (equation (5.2.2)), and $IC_{BTC,ETH}$ (equation (5.2.3)), all constructed for maturity of 30 calendar days. We report the specifications with one, two, and three independent variables. The data are sampled daily from 05-2021 to 11-2023.

### 6.3 Bear Markets and Implied Variables

Next, we investigate the dynamics of our measures in bear markets. Interestingly, the two worst days for BTC and ETH over our sample were on the 14th of July 2022 and the 10th of November 2022 when BTC experienced a negative return of -0.16 and -0.14, and ETH -0.18 and -0.17. Figure 6.5 displays the behavior of the variables around these days. Comparing Panel (a) and Panel (b) it turns out that the $IL^Q$ steeply increased on the second event while on the first event, it did react delayed. For the $IV_{BTC}$ a rapid increase is visible in both figures (Panel (c) and Panel (d)) and the same is true and even more pronounced for $IV_{ETH}$, as depicted on Panel (e) and Panel (f). Lastly, the strong negative return in both tokens increases the implied correlation (Panel (g)) and Panel (h)).

\(^7\)We have carried out the same exercise for changes in the implied impermanent loss and its components. Doing so leads to comparable results, with worse explanatory power.
Figure 6.5: Option Implied Variables and Worst Days of both Underlyings. The figure displays the implied variables around the worst days of the underlying in our sample, which was 2022-06-14 and 2022-11-10 where BTC experienced a drawdown of -0.16 and -0.14, and ETH -0.18 and -0.17 respectively. The implied impermanent loss (equation (5.2.1)) and its components, namely, the $IV_{BTC}$, $IV_{ETH}$ (equation (5.2.2)), and $IC_{BTC,ETH}$ (equation (5.2.3)), all constructed for maturity of 30 calendar days.
Next, we investigate the dynamics of our measures for days when one token experiences a large decline while the other does not. For example, BTC experienced a large drawdown on the 5th of December 2021 with a negative return of $-0.08$ while ETH only experienced a small drawdown of $-0.02$. Adding to that, on the 16th of September 2022 ETH experienced a large negative drawdown of $-0.10$ while BTC only $-0.02$.

Figure 6.6 displays the $IL^Q$ and the $IC$ on these two days for BTC and ETH: As visible the $IL^Q$ responds before and during the drawdown. The effect is stronger for ETH (Panel (b)). In line with the intuition, the $IC$ drops sharply in both events (Panel (c) and Panel (d)).

![Figure 6.6: Option Implied Variables and Worst Days for each Underlyings.](https://ssrn.com/abstract=4811111)

6.4 Cross-Sectional Implications for Liquidity Provision

To check if there is a risk-return relationship between the impermanent loss and the returns of the liquidity provision, a two-stage Fama and MacBeth (1973) regression is performed, where changes in the realized impermanent loss and in the implied variables are considered risk factors. As test assets, the APRs from the Uniswap liquidity pools are utilized, where their betas are estimated over a one-year rolling estimation window. Hence in the first stage, the APRs are regressed on a constant and the changes in the implied variables. In the second stage, the expected APRs are regressed on these
betas from the first stage. To proxy for expected APRs we rely on the realized returns over the next 1, 2, 3, 5, 10, and 14 days. The coefficients from the second stage regression ($\gamma$) represent the average risk premia of the risk factors.

6.4.1 Univariate Analysis

First, we test the ability to price the cross-section of liquidity pools using the daily and monthly realized impermanent loss. As displayed in Figure 6.7 Panel (b) the associated t-stat of the second-stage regression coefficient is only significant for the next day’s APR. As visible from Panel (a) and Panel (c), the price of risk for the realized impermanent loss is positive. In contrast, the implied impermanent loss, where the forward-looking information of the options is encapsulated, displays significance for up to 7 days.

Figure 6.7: Cross-Section – Univariate – Impermanent Loss. The table reports the $\gamma$ coefficient and its t-stat for the Fama and MacBeth (1973) two-stage cross-sectional regression. As risk factors, changes in the daily realized impermanent loss ($IL_P^t$ (equation (5.1))), and changes in the implied impermanent loss (equation (5.2.1)) are considered. The implied variables are constructed for a maturity of 30 calendar days. The sample period is from 05-2021 to 11-2023.

Next, we repeat the procedure for the implied drivers of the implied impermanent loss, that is, the individual volatilities and their implied correlation. Figure 6.8 displays the results: The $IV$s are positively priced and significant for up to 10 days (see Panel (a), (b), (c), and (d)). For implied correlation, the regression coefficient is negative and statistically significant up to 7 days (Panel (e)).

The monthly realized impermanent loss exhibits significance for up to 3 days, see Figure C.3.
and Panel (f)). In liquidity provision, LPs prefer tokens with high positive correlations to minimize impermanent loss. As a result, the price of risk, with implied correlation as a factor, is negative.

\[(a) \Delta IV_{BTC}: \gamma\]

\[(b) \Delta IV_{BTC}: \text{t-stat}\]

\[(c) \Delta IV_{ETH}: \gamma\]

\[(d) \Delta IV_{ETH}: \text{t-stat}\]

\[(e) \Delta IC_{BTC,ETH}: \gamma\]

\[(f) \Delta IC_{BTC,ETH}: \text{t-stat}\]

Figure 6.8: Cross-Section – Univariate – Drivers of the Implied Impermanent Loss. The table reports the \(\gamma\) coefficient and its t-stat for the Fama and MacBeth (1973) two-stage cross-sectional regression. As risk factors, the drivers of the implied impermanent loss are considered, namely changes in the \(IV_{BTC}\), \(IV_{ETH}\) (equation (5.2.2)), and changes in \(IC_{BTC,ETH}\) (equation (5.2.3)), all constructed for maturity of 30 calendar days. The sample period is from 05-2021 to 11-2023.

6.4.2 Multivariate Analysis

We repeat the procedure for a multivariate setting where we simultaneously estimate the betas for more than one factor. Figure 6.9 displays the \(\gamma\) and its t-stats over the predictive horizons. In line with the univariate regressions, the volatilities (correlation) carry a positive (negative) price of risk. The synergy among the implied variables boosts the predictive horizon for up to 10 days and for the
correlation up to 14 days (10% confidence level).

Figure 6.9: Cross-Section – Multivariate – Drivers of the Implied Impermanent Loss. The table reports the $\gamma$ coefficient and its t-stat for the multivariate Fama and MacBeth (1973) two-stage cross-sectional regression. As risk factors, the drivers of the implied impermanent loss are considered, namely changes in the $IV_{BTC}$, $IV_{ETH}$ (equation (5.2.2)), and changes in $IC_{BTC,ETH}$ (equation (5.2.3)), all constructed for maturity of 30 calendar days. The sample period is from 05-2021 to 11-2023.
7 Robustness

To verify the robustness results of the analysis to various specifications, a series of tests are carried out and reported in the Appendix.

8 Conclusion

The predictive power of the implied impermanent loss, which reflects the risk of a disparity in relative token prices as implied by option prices, remains robust in forecasting liquidity provision returns. This paper delves deeper into its predictive capabilities and, crucially, investigates its fundamental economic drivers.

We assess the impermanent loss from a risk-neutral standpoint. Impermanent loss equates to one-eighth of the volatility of the relative price and can be computed by evaluating the log contract Carr and Madan (1999) using a portfolio of European options on the relative price. To tackle the primary challenges—namely, the lack of an options market for the relative price and the deviation of the relative price from a martingale process—we compute a joint density, minimizing the Hansen and Jagannathan (1991) bounds, from existing options on separate tokens to address the first difficulty. To handle the second challenge, we employ a change of numéraire. We therefore also make a methodological contribution.

We present empirical findings indicating that the predictability of implied impermanent loss stems from its three primary components: implied idiosyncratic token variances and implied diversification risk. These components individually account for the variation in liquidity pool returns across different tokens. Interestingly, combinations of these components often yield better predictions of the cross-section than implied impermanent loss itself, suggesting that some valuable information may be lost when consolidating them into the specific functional form of implied impermanent loss.

References


Demeterfi, Derman, Kamal, and Zou. More than you ever wanted to know about volatility swaps. 1999.


A Stochastic Differential Equations

A.1 Proof of Proposition 3.1 for Impermanent Loss

By the relative price that is defined in equation (3.2) and Assumption 3.1, we have \( \frac{N_2(t)}{N_1(t)} = R(t) = \frac{P_1(t)}{P_2(t)} \). Based on equations (3.3), (3.5) and (3.6), using Itô’s lemma we show that

\[
d\left( N^1(t)P^1(t) + N^2(t)P^2(t) \right) - \left( N^1(t)dP^1(t) + N^2(t)dP^2(t) \right) = -\frac{\tilde{\sigma}^2(t)}{4}N^1(t)P^1(t)dt .
\]

This shows that the numerator in equation (3.8) has only a dt term, and so we take the limit in the denominator of equation (3.8) as \( \Delta t \) goes to zero, giving us

\[
d\text{IL}(t) = -\frac{\tilde{\sigma}^2(t)N^1(t)P^1(t)dt}{4(N^1(t)P^1(t) + N^2(t)P^2(t))} . \tag{A.1}
\]

Then again, from equation (3.2) and Assumption 3.1 we have \( \frac{N_2(t)}{N_1(t)} = 1 \), and equation (A.1) reduces to equation (3.10).

A.2 Change of Numéraire and Girsanov Theorem

Recall our pair of SDEs for token prices,

\[
dP^1(t) = rP^1(t)dt + \sigma_1(t)P^1(t)dB^{1,Q}(t),
\]

\[
dP^2(t) = rP^2(t)dt + \sigma_2(t)P^2(t)dB^{2,Q}(t),
\]

where \( B^{1,Q}(t) \) and \( B^{2,Q}(t) \) are risk-neutral standard Brownian motions with correlation parameter \( \rho \), with \( \mathbb{Q} \) denoting the risk-neutral measure and \( \mathbb{E}^{\mathbb{Q}} \) denoting risk-neutral expectation.

\[
\text{Call}^\text{pr} (T, K) = e^{-rT}\mathbb{E}^{\mathbb{Q}}(P^1(T) - KP^2(T))^+
\]

\[
= e^{-rT}\mathbb{E}^{\mathbb{Q}} (P^2(T)(R(T) - K)^+)
\]

\[
= P^2(0)\mathbb{E}^{\mathbb{Q}} \left( \frac{P^2(T)}{P^2(0)e^{rT}}(R(T) - K)^+ \right)
\]

\[
= P^2(0)\tilde{\mathbb{E}}^{\mathbb{Q}}(R(T) - K)^+ ,
\]

where \( \frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}} \bigg|_{T} = \frac{P^2(T)}{P^2(0)e^{rT}} \), under which \( R(t) = P^1(t)/P^2(t) \) is a \( \tilde{\mathbb{Q}} \) martingale.

\[
\tilde{\mathbb{E}}^{\mathbb{Q}} R(T) = \mathbb{E}^{\mathbb{Q}} \left[ \frac{P^2(T)}{P^2(0)e^{rT}} \frac{P^1(T)}{P^2(T)} \right] = \mathbb{E}^{\mathbb{Q}} \left[ \frac{P^1(T)}{P^2(0)e^{rT}} \right] = \frac{P^1(0)e^{rT}}{P^2(0)e^{rT}} = R(0) .
\]
Let
\[ X(t) = \int_0^t \sigma_2(s) dB^{2, Q}(s), \]
which is a martingale under \( Q \). The Doleans-Dade exponent is
\[ \frac{d\tilde{Q}}{dQ} \big|_T = E(X(T)) = e^{X(T) - \frac{1}{2} \sigma^2(t) dt} = \frac{P^2(T)}{P^2(0) e^{rT}}. \]

Girsanov Theorem is the following: Given \( W(t) \) that is Brownian motion under \( Q \), we define
\[ \tilde{W}(t) = W(t) - [W, X](t) \]
where \([\cdot, \cdot]\) denotes quadratic cross variation, and then \( \tilde{W}(t) \) is Brownian motion under \( \tilde{Q} \). In particular, \( \tilde{B}^{2, Q}(t) = B^{2, Q}(t) - \int_0^t \sigma_2(s) ds \) is \( \tilde{Q} \) Brownian motion. Suppose for \( Q \) Brownian motion \( W(t) \) that \( dB^{2}(t) dW(t) = \rho dt \). Then by Girsanov theorem, define
\[ \tilde{W}(t) = W(t) - [W, X](t) = W(t) - \rho \int_0^t \sigma_2(s) ds, \]
which is \( \tilde{Q} \) Brownian motion. In particular, for the token prices \( P^1(t) \) and \( P^2(t) \), under the new measure we have
\[ dP^1(t) = rP^1(t) dt + \sigma_1(t) P^1(t) \left( dB^{1, Q}(t) - \rho \sigma_2(t) dt + \rho \sigma_2(t) dt \right) \]
\[ = (r + \rho \sigma_1(t) \sigma_2(t)) P^1(t) dt + \sigma_1(t) P^1(t) dB^{1, Q}(t) \]
\[ dP^2(t) = (r + \sigma_2^2(t)) P^2(t) dt + \sigma_2(t) P^2(t) dB^{2, Q}(t) \]
where \( B^{1, Q}(t) = B^{1, Q}(t) - \rho \int_0^t \sigma_2(s) ds \). Notice that \( e^{-rt}/P^2(t) \) is a \( \tilde{Q} \) martingale (i.e., the discounted price of the original ‘currency’ is a martingale under the change of numéraire).

Variance Swaps and Impermanent Loss

Assuming constant interest rate and zero dividend yield, the variance swap rate (Bakshi et al. (2015); Carr and Madan (1999); Demeterfi et al. (1999)) is
\[ E^Q \left( \frac{1}{T} \int_0^T \sigma_i^2(t) dt \right) = \frac{2}{T} E^Q \left( \int_0^T \frac{dP^i(t)}{P^i(t)} \right) - \frac{2}{T} E^Q \log \left( \frac{P^i(T)}{P^i(0)} \right) \]
\[ = \frac{2}{T} E^Q \log \left( \frac{P^i(T)}{P^i(0) e^{rT}} \right) \quad \text{for } i = 1, 2. \]
Then for the log-contracts on $P^1(T)$ and $P^2(T)$ under tilde measure,

$$
\frac{2}{T} \tilde{\mathbb{E}}^Q \log \left( \frac{P^1(T)}{P^1(0)} e^{\sigma_1(T)} \right) = \frac{2}{T} \tilde{\mathbb{E}}^Q \int_0^T \left( \rho \sigma_1(t) \sigma_2(t) - \frac{1}{2} \sigma_1^2(t) \right) dt
$$

$$
\frac{2}{T} \tilde{\mathbb{E}}^Q \log \left( \frac{P^2(T)}{P^2(0)} e^{\sigma_1(T)} \right) = \frac{1}{T} \tilde{\mathbb{E}}^Q \int_0^T \sigma_2^2(t) dt .
$$

Combining these last two equations we obtain a tilde-measure expectation for impermanent loss,

$$
\tilde{\mathbb{E}}^Q IL(T) = \frac{1}{8T} \tilde{\mathbb{E}}^Q \int_0^T \bar{\sigma}^2(t) dt = \frac{2}{T} \tilde{\mathbb{E}}^Q \log \left( \frac{P^2(T)}{P^1(T)} \right) = -2\tilde{\mathbb{E}}^Q R(T) ,
$$

where $R(t) = P^1(t)/P^2(t)$ and $\bar{\sigma}^2(t) = \sigma_1^2(t) + \sigma_2^2(t) - 2\rho \sigma_1(t) \sigma_1(t)$.

**B Quadratic Program for Hansen-Jagannathan Bound**

Consider random variables $X \in \{x_1, x_2, \ldots, x_n\}$ and $Y \in \{y_1, \ldots, y_m\}$, with given joint real-world distribution

$$
p_{ij} = \mathbb{P}(X = x_i, Y = y_j) .
$$

Let $\Pi$ denote the excess return on a portfolio, and assume that $(X, Y)$ is the state variable. Let $\mathbb{E}$ denote expectation and let $\sigma$ denote standard deviation, both under the physical measure. Let $M$ denote a pricing kernel such that

$$
M_{ij} = \frac{q_{ij}}{p_{ij}} ,
$$

where $q_{ij}$ is an equivalent risk-neutral measure. For any $R$ and $q$, we have

$$
0 = \mathbb{E}[\Pi M] = \text{cov}(\Pi M) + \mathbb{E}\Pi \geq -\text{std}(\Pi)\text{std}(M) + \mathbb{E}\Pi ,
$$

which yields the Hansen-Jagannathan (HJ) bound on Sharpe Ratios,

$$
\sup_{\Pi: \text{std}(\Pi) > 0} \frac{\mathbb{E}\Pi}{\text{std}(\pi)} \leq \inf_{M \in \mathcal{M}} \text{std}(M) ,
$$

where $\mathcal{M}$ denotes the family of pricing kernels. The right-hand side motivates us to find a pricing kernel that minimizes the HJ bound.
B.1 The Unconstrained Case

For risk-neutral distribution $q$ let $\phi_{ij} := q_{ij}/\sqrt{p_{ij}}$, so that $\text{var}(M) = \sum_{i,j} \left( \frac{q_{ij}}{p_{ij}} - 1 \right)^2 p_{ij} = \sum_{i,j} \phi_{ij}^2 - 1$. Let's start with an unconstrained problem

$$\min_{\phi: \phi_{ij} > 0} \sum_{i,j} \phi_{ij}^2 ,$$

s.t.

$$\sum_{i,j} \phi_{i,j} \sqrt{p_{i,j}} = 1 .$$

There is solution to this quadratic program (QP). In matrix/vector form, we write this as

$$\min_{\tilde{\phi}} \tilde{\phi}^T \tilde{\phi} ,$$

s.t.

$$\tilde{\phi}^T \sqrt{\tilde{p}} = 1$$

$$\tilde{\phi}_i \geq 0 \quad \forall i \leq nm ,$$

where $\tilde{\phi}$ stacks the columns of $\phi$ into an $nm$ vector,

$$\tilde{\phi} = \begin{bmatrix} \phi_{11} \\ \phi_{21} \\ \vdots \\ \phi_{n1} \\ \phi_{12} \\ \phi_{22} \\ \vdots \\ \vdots \\ \phi_{nm} \end{bmatrix} ,$$

and where $\sqrt{\tilde{p}}$ denotes the $nm$ stacked vectorization of $p$ with the square-root function applied entry-wise. This matrix/vector form of this QP can be solved with standard software packages.
B.2 Constrained for Marginals

Suppose we have the risk-neutral marginals given, $\sum_{j} q_{ij} = \mu_i$ and $\sum_{i} q_{ij} = \nu_j$. We then write a constrained problem

$$\min_{\phi_{ij} > 0} \sum_{i,j} \phi_{ij}^2,$$

s.t.

$$\sum_{j} \phi_{i,j} \sqrt{p_{i,j}} = \mu_i$$

$$\sum_{i} \phi_{i,j} \sqrt{p_{i,j}} = \nu_j.$$ 

For each $i \leq n$ and each $j \leq n$ denote

$$A_i = I_{m \times m} \otimes e_i e_i^T$$

$$B_j = \text{diag}(e_j) \otimes I_{n \times n},$$

where $\otimes$ is the Kronecker product, and where $e_i \in \mathbb{R}^n$ and $e_j \in \mathbb{R}^m$ are, respectively, the $i^{th}$ and $j^{th}$ canonical basis vector vectors. To gain a sense for the operations performed by $A_i$ and $B_i$, take for example the following multiplications,

$$A_i \vec{\phi} = \begin{bmatrix} e_i e_i^T & \cdots & \cdots & e_i e_i^T \end{bmatrix} \vec{\phi} = \begin{bmatrix} \phi_{i1} \\ 0 \\ \vdots \\ 0 \\ \phi_{i2} \\ 0 \\ \vdots \\ \vdots \\ 0 \\ \phi_{im} \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \quad \forall i \leq n,$$
and

$$B_j \vec{\phi} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ I_{n \times n} \\ \vdots \\ 0 \\ 0 \end{bmatrix} \vec{\phi} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \phi_{1j} \\ \vdots \\ \phi_{nj} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \forall j \leq m.$$ 

These are symmetric matrices, \( A_i = A_i^T \) and \( B_i = B_i^T \), and are also idempotent,

\[ A_i^2 = A_i \quad B_i^2 = B_i. \]

Thus, \( A_i \) and \( B_i \) can be used to write the constraints,

\[ \vec{\phi}^T A_i \sqrt{\vec{p}} = \sum_j \phi_{i,j} \sqrt{p_{i,j}} = \mu_i \]
\[ \vec{\phi}^T B_j \sqrt{\vec{p}} = \sum_i \phi_{i,j} \sqrt{p_{i,j}} = \nu_j. \]

Next, denote \( v_i = A_i \sqrt{\vec{p}} \) and \( \ell_j = B_j \sqrt{\vec{p}} \). Now the QP can be nicely written in matrix/vector form,

\[ \min_{\vec{\phi}} \vec{\phi}^T \vec{\phi}, \]
\[ \text{s.t.} \]
\[ v_i^T \vec{\phi} = \mu_i \quad \forall i \leq n \]
\[ \ell_j^T \vec{\phi} = \nu_j \quad \forall j \leq m \]
\[ \vec{\phi}_i \geq 0 \quad \forall i \leq nm. \]
Here, $v_i \in \mathbb{R}^{nm}$ is $m$-sparse and $\ell_j \in \mathbb{R}^{nm}$ is $n$-sparse. These sparse vectors can be loaded into a single constraint matrix $C \in \mathbb{R}^{(m+n) \times mn}$ that is $2mn$-sparse,

$$
C = \begin{bmatrix}
v_1^T \\
v_2^T \\
\vdots \\
v_n^T \\
\ell_1^T \\
\ell_2^T \\
\vdots \\
\ell_m^T 
\end{bmatrix}.
$$

To save time in computing, the entries of matrix $C$ can be loaded prior to the running of optimization routines. Using $C$, we can write the QP as simply

$$
\min_{\vec{\phi}} \vec{\phi}^T \vec{\phi}, \\
\text{s.t.} \\
C\vec{\phi} = \begin{bmatrix} \mu \\ \nu \end{bmatrix}, \\
\vec{\phi}_i \geq 0 \quad \forall i \leq nm.
$$

**Implementation**

We need a software package for doing QPs with $(m+n)$-many sparse constraints. This problem quickly blows up; for example take $n = m = 100$, which has 200 constraints each with a $10000 \times 10000$ sparse matrix. Matlab appears to offer a software package for sparse QPs.
Figure C.1: Option Volumes across Exchanges. The figure reports the Bitcoin (Panel (a)) and Ethereum (Panel (b)) options trading volume, in dollar terms, across cryptocurrency exchanges. The data is obtained from The Block.
Figure C.2: IV – BTC and ETH. The figure shows the implied impermanent loss ($IL^Q$) and realized impermanent loss ($IL^P$) for BTC and ETH for different moving averages (7 days in Panel (a), and 2 days in Panel (b)). The implied IL is calculated following equation (5.2.1) and for a maturity of 30 days. The daily realized impermanent loss is calculated using equation (5.1), annualized, and expressed in terms of volatility. The data is sampled daily, and the sample period is from 05-2021 to 11-2023.

Figure C.3: Cross-Section – Univariate – Impermanent Loss. The table reports the $\gamma$ coefficient and its t-stat for the Fama and MacBeth (1973) two-stage cross-sectional regression. As risk factor, changes in the daily 30-day realized impermanent loss ($IL^P_{30}$) (equation (5.1)) are considered. The sample period is from 05-2021 to 11-2023.